

Local Well-Posedness of Dynamics of Viscous Gaseous Stars

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February 1, 2008

Abstract

We establish the local in time well-posedness of strong solutions to the vacuum free boundary problem of the compressible Navier-Stokes-Poisson system in the spherically symmetric and isentropic motion. Our result captures the physical vacuum boundary behavior of the Lane-Emden star configurations for all adiabatic exponents $\gamma > \frac{6}{5}$.

1 Formulation and Notation

The motion of self-gravitating viscous gaseous stars can be described by the compressible Navier-Stokes-Poisson system:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0, \\ \frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p &= -\rho \nabla \Phi + \mu \Delta \mathbf{u}, \\ \Delta \Phi &= 4\pi \rho, \end{aligned} \tag{1.1}$$

where $t \geq 0$, $\mathbf{x} \in \mathbb{R}^3$, $\rho \geq 0$ is the density, $\mathbf{u} \in \mathbb{R}^3$ the velocity, p the pressure of the gas, Φ the potential function of the self-gravitational force, and $\mu > 0$ the constant viscosity coefficient. We consider polytropic gases and the equation of state is given by

$$p = A\rho^\gamma$$

where A is an entropy constant and $\gamma > 1$ is an adiabatic exponent; in this case, the motion is called barotropic, which means the pressure does not depend on the temperature or specific entropy. Values of γ have their own physical significance [3]; for example, $\gamma = \frac{5}{3}$ stands for monatomic gas, $\frac{7}{5}$ for diatomic gas, $\gamma \searrow 1$ for heavier molecules. These γ 's also take important part in the existence, uniqueness, and stability of stationary solutions, for instance, see [1, 5, 6] for inviscid gaseous stars modeled by the Euler-Poisson system.

For the spherically symmetric motion, i.e. $\rho(t, \mathbf{x}) = \rho(t, r)$ and $\mathbf{u}(t, \mathbf{x}) = u(t, r)\frac{\mathbf{x}}{r}$, where u is a scalar function and $r = |\mathbf{x}|$, (1.1) can be written as follows:

$$\begin{aligned} \rho_t + \frac{1}{r^2}(r^2 \rho u)_r &= 0, \\ \rho u_t + \rho u u_r + p_r + \frac{4\pi \rho}{r^2} \int_0^r \rho s^2 ds &= \mu(u_{rr} + \frac{2u_r}{r} - \frac{2u}{r^2}). \end{aligned} \tag{1.2}$$

Stationary solutions $\rho = \rho_0(r)$ and $u = 0$, non-moving gaseous spheres, satisfy the following:

$$(p_0)_r + \frac{4\pi \rho_0}{r^2} \int_0^r \rho_0 s^2 ds = 0. \tag{1.3}$$

Note that the viscosity has nothing to do with these static solutions themselves, namely, they are also stationary solutions of the Euler-Poisson system [5]. The ordinary differential equation (1.3) can be transformed into the famous Lane-Emden equation, and solutions of (1.3) can be characterized according to γ in the following fashion: for given finite total mass, if $\gamma \in (\frac{6}{5}, 2)$, then there exists at least one compactly supported stationary solution ρ_0 . For $\gamma \in (\frac{4}{3}, 2)$, every stationary solution is compactly supported and unique. If $\gamma = \frac{6}{5}$, there is a unique solution ρ with infinite support, and it can be written explicitly in terms of the Lane-Emden function. On the other hand, if $\gamma \in (1, \frac{6}{5})$, there are no stationary solutions with finite total mass. For $\gamma \in (\frac{6}{5}, 2)$, letting $r = R$ be the finite vacuum boundary of steady stars, it is well known [6] that

$$\rho_0(r) \sim (R - r)^{\frac{1}{\gamma-1}} \quad \text{if } r \sim R. \quad (1.4)$$

We observe that for given finite total mass, the maximum value of density ρ_0 is inversely proportional to R the radius of stars. For the details of stationary solutions such as the existence and the asymptotic behavior, according to γ , we refer to [6].

Our main interest is the evolution of stars with finite radii as a free boundary problem with the vacuum boundary to (1.2). Let $r \in [0, R(t)]$ and $t \in [0, T]$ for $T > 0$. We look for $\rho(t, r)$ and $R(t)$ so that

$$\rho > 0 \text{ for } r < R(t) \text{ and } \rho(t, R(t)) = 0; \quad (1.5)$$

in particular, we would like to capture the behavior of interesting stationary density profiles (1.4) near the vacuum boundary $r = R$. On the free boundary $r = R(t)$, we impose the kinematic boundary condition

$$\frac{d}{dt}R(t) = u(t, R(t)), \quad (1.6)$$

and the dynamic boundary condition

$$(\mu u_r - p)(t, R(t)) = 0. \quad (1.7)$$

We remark that $\rho(t, R(t)) = 0$ also serves the boundary condition in our vacuum problem; in order to see this, we formally compute the rate of density change in t along the particle path $R(t)$:

$$\begin{aligned} \frac{d}{dt}\rho(t, R(t)) &= \rho_t(t, R(t)) + (\rho_r u)(t, R(t)) = -\rho(t, R(t)) \frac{(r^2 u)_r}{r^2}(t, R(t)) \\ &\Rightarrow \rho(t, R(t)) = \rho(0, R(0)) \exp\left\{-\int_0^t \left(u_r + \frac{2u}{r}\right)(\tau, R(\tau)) d\tau\right\} \\ &\text{or } \rho(t, R(t))R(t)^2 = \rho(0, R(0))R_0^2 \exp\left\{-\int_0^t u_r(\tau, R(\tau)) d\tau\right\}. \end{aligned}$$

Thus $\rho(t, R(t)) = 0$ for all time if it vanishes initially. Note that the dynamic boundary condition with the vacuum boundary condition leads to Neumann boundary condition

$$u_r(t, R(t)) = 0.$$

Due to lack of current mathematical tools to deal with the vacuum boundary in Eulerian coordinates, it is desirable to introduce Lagrangian formulation of (1.2). We may assume that the total mass of stars is 4π , since the total mass is preserved in time:

$$\begin{aligned} \frac{d}{dt} \int_0^{R(t)} \rho(t, r) r^2 dr &= \rho(t, R(t)) R^2(t) \frac{d}{dt} R(t) + \int_0^{R(t)} \rho_t(t, r) r^2 dr \\ &= \rho(t, R(t)) R^2(t) u(t, R(t)) - R^2(t) \rho(t, R(t)) u(t, R(t)) \\ &= 0 \end{aligned}$$

where we have used the boundary condition (1.6) and the continuity equation at the second equality.

Now we introduce a new independent variable x :

$$x \equiv \int_0^r \rho s^2 ds, \quad (1.8)$$

which is a Lagrangian (mass) variable. The domain of x is $[0, 1]$, since the total mass is assumed to be 4π . Denote Lagrangian derivatives by D_t, D_x . By change of variables, the Lagrangian mass coordinate system (t, x) and the Eulerian coordinates (t, r) obey the following relation:

$$D_t = \partial_t + (D_t r) \partial_r, \quad D_x = (D_x r) \partial_r. \quad (1.9)$$

In Lagrangian coordinates, r is not an independent variable, but a function of t, x . To investigate the dynamics of r , first fix $x = x_0$. Then $r = r(t, x_0) \equiv r(t)$ is a particle path, and by taking $\frac{d}{dt}$ of (1.8) and by the continuity equation, we get

$$0 = \rho(t, r(t)) r^2(t) (D_t r) + \int_0^{r(t)} \partial_t \rho s^2 ds = \rho(t, r(t)) r^2(t) (D_t r(t)) - r^2(t) \rho(t, r(t)) u(t, r(t))$$

and hence $D_t r = u$. Since $\partial_r x = \rho r^2$ from (1.8), by using the inverse function differentiation, we obtain $D_x r = \frac{1}{\rho r^2}$. We therefore conclude the dynamics of r in Lagrangian formulation as the following:

$$D_t r = u, \quad D_x r = \frac{1}{\rho r^2}. \quad (1.10)$$

The second relation formally leads to

$$r = \left\{ 3 \int_0^x \frac{1}{\rho} dy \right\}^{\frac{1}{3}}. \quad (1.11)$$

We notice that the dynamics of ρ, u completely determines r . As in (1.9), ∂_t, ∂_r can be expressed in terms of D_t, D_x :

$$\partial_t = D_t - r^2 \rho u D_x, \quad \partial_r = \rho r^2 D_x.$$

By using this change of variables with (1.10), one can easily check that (1.2) can be written in Lagrangian coordinates (t, x) as follows: for $0 \leq x \leq 1$,

$$\begin{aligned} D_t \rho + \rho^2 r^2 D_x u + \frac{2\rho u}{r} &= 0, \\ D_t u + r^2 D_x p + \frac{4\pi x}{r^2} + \mu \frac{2u}{\rho r^2} &= \mu D_x (\rho r^4 D_x u), \\ \text{or } D_t u + r^2 D_x p + \frac{4\pi x}{r^2} &= \mu r^2 D_x (\rho r^2 D_x u + \frac{2u}{r}), \end{aligned} \quad (1.12)$$

with the boundary conditions

$$u(t, 0) = 0, \quad (\mu \rho r^2 D_x u - p)(t, 1) = 0, \quad \text{and } \rho(t, 1) = 0. \quad (1.13)$$

We point out that $D_t \equiv \partial_t + u \partial_r$ represents the material derivative in the spherically symmetric motion. The vacuum boundary condition also yields $(\rho r^2 D_x u)(t, 1) = 0$, but note that we cannot reduce to $D_x u = 0$ due to the vanishing property of ρ . A free boundary $R(t)$ corresponds to a fixed boundary $x = 1$ in Lagrangian formulation. It is easy to check that Jacobian of its transformation is ρr^2 and hence two formulations are equivalent as long as $\rho > 0$ and $r > 0$. Note that u needs not to be zero along $x = 1$.

Let $\rho_0 = \rho_0(r)$ be the given stationary profile in Eulerian coordinates as a solution of (1.3) with the total mass 4π . The decay rate of ρ_0 towards $x = 1$ in Lagrangian coordinates is asymptotically given as the following:

$$\rho_0 \sim (1 - x)^{\frac{1}{7}} \quad \text{if } x \sim 1 \quad (1.14)$$

from (1.8) and (1.4).

Variations of this model have been considered in the literature: Okada and Makino [9, 10] studied global weak solution, uniqueness, and stability to the Navier-Stokes equations for gas surrounding a solid ball (a hard core) without self-gravitation in Lagrangian coordinates as a free boundary problem when the density distribution contacts with the vacuum at a finite radius. More recently, a similar study has been done by Ducomet and Zlotnik [2] for viscous compressible flow driven by gravitation and an outer pressure, proving interesting stabilization results. However, all their results are restricted to cutoff domains excluding some neighborhood of the origin, since their analysis strongly depends on the one-dimensional structure of symmetric flows. On the other hand, when the initial density is away from the vacuum for smooth initial data or discontinuous data, one-dimensional or multidimensional problem, a lot of progress has been made on the compressible Navier-Stokes equations. However, as far as the physical vacuum is concerned, very few rigorous results are available for compressible flows. For one-dimensional viscous gas flow, there has been some important progress; in particular, the vacuum interface behavior as well as the regularity to one-dimensional Navier-Stokes free boundary problems were investigated by Luo, Xin, and Yang [7]. It is important and interesting to understand the dynamics of the Navier-Stokes-Poisson system in three space dimension as a vacuum free boundary problem displaying the feature of physical vacuum boundary behavior (1.4) or (1.14).

This article concerns the local well-posedness of a free boundary problem to the Navier-Stokes-Poisson system (1.2) with (1.5), (1.6), and (1.7), or (1.12) with (1.13), embracing the physical vacuum boundary behavior (1.4) or (1.14) of stationary density profiles, especially when $\frac{6}{5} < \gamma < 2$. To establish local in time strong solutions including the physical vacuum boundary, we utilize both Eulerian and Lagrangian formulations.

Before we state the main results, we first define the suitable energy space in which regular solutions reside. Let the initial density profile be given by ρ_{in} satisfying the following conditions:

$$\begin{aligned} & \text{(i) } \rho_{in} > 0 \text{ for } r < R \text{ and } \rho_{in}(R) = 0, \\ & \text{(ii) } \int_0^R \rho_{in} s^2 ds = 1 \text{ (total mass} = 4\pi). \end{aligned} \quad (1.15)$$

Consider $0 < r_0 < r_1 < r_2 < R$ such that

$$0 < 2d < r_0, \quad 0 < 3d < r_2 - r_1, \quad \frac{1}{r_0 - d} \leq 1, \quad (1.16)$$

for small fixed constant d . Now let x_i be the initial position in Lagrangian coordinates corresponding to r_i where $i = 0, 1, 2$:

$$x_i = \int_0^{r_i} \rho_{in} s^2 ds. \quad (1.17)$$

Then by the positivity of ρ_{in} , we get $0 < x_0 < x_1 < x_2 < 1$. Denoting the particle path emanating from r_i by $r_i(t)$, $r_i(t)$ characterizes x_i :

$$\frac{d}{dt} \int_0^{r_i(t)} \rho(s, t) s^2 ds = 0, \text{ i.e. } \int_0^{r_i(t)} \rho(s, t) s^2 ds = x_i. \quad (1.18)$$

This relation is the conservation of mass and can be readily verified by using the continuity equation and

$$\frac{d}{dt} r(t) = u(r(t), t).$$

Assume $|u(r, t)| \leq K$ for all $0 \leq r \leq R(t)$ and $0 \leq t \leq T$, where T is sufficiently small. In particular, d in (1.16) will be chosen so that $KT \leq d$. Note that the smallness assumption on the time interval T prevents a dramatic change of r in time.

We note that the initial data ρ_{in} and u_{in} given in Eulerian coordinates can be also regarded as functions of x in Lagrangian coordinates, since (1.8) is valid when $t = 0$: $x = \int_0^r \rho_{in} s^2 ds$. The initial

value of r in Lagrangian coordinates is given or defined as the following: $r_{in}(x) = (3 \int_0^x \frac{1}{\rho_{in}} dy)^{\frac{1}{3}}$ from (1.11). As the first preparation of the rigorous analysis, we introduce the following cutoff functions χ and ζ . Let $\chi \in \mathcal{C}^\infty[0, 1]$ be a smooth function of x such that

- (i) $0 \leq \chi \leq 1$ and $\text{supp}(\chi) \subset [x_0, 1]$,
- (ii) $\chi(x) = 1$ if $x_1 \leq x \leq 1$,
- (iii) $|\chi'| \leq \frac{C}{x_1 - x_0}$ and $|\chi''| < \infty$.

χ is a function of both r and t in Eulerian coordinates. Note that

$$|r_i(t) - r_i| \leq d \text{ for } 0 \leq t \leq T,$$

since

$$r_i(t) = r_i + \int_0^t u(r(\tau), \tau) d\tau \text{ by (1.10).}$$

Hence we deduce that for $0 \leq t \leq T$,

$$\chi(r, t) = 0 \text{ if } r \leq r_0 - d \text{ and } \chi(r, t) = 1 \text{ if } r \geq r_1 + d.$$

Similarly, construct a smooth function ζ of r such that

- (i) $0 \leq \zeta \leq 1$ and $\text{supp}(\zeta) \subset [0, r_2 - d]$,
- (ii) $\zeta(r) = 1$ if $0 \leq r \leq r_1 + d$,
- (iii) $|\zeta'| \leq \frac{C}{r_2 - r_1 - 2d}$ and $|\zeta''| < \infty$.

Then as a function of x and t , ζ satisfies the following: for $0 \leq t \leq T$,

$$\zeta(x, t) = 1 \text{ if } x \leq x_1 \text{ and } \zeta(x, t) = 0 \text{ if } x \geq x_2.$$

We will freely view χ and ζ as functions of x , r and t without confusion.

Define the energy functional $\mathcal{E}(t)$ by the sum of the Eulerian energy $\mathcal{E}_E(t)$ and the Lagrangian energy $\mathcal{E}_L(t)$:

$$\mathcal{E}(t) = \mathcal{E}_E(t) + \mathcal{E}_L(t)$$

where

$$\mathcal{E}_E(t) \equiv \frac{1}{2} \sum_{|\alpha| \leq 3} \{A\gamma \int \zeta \rho^{\gamma-2} |\partial^\alpha \rho|^2 d\mathbf{x} + \int \zeta \rho |\partial^\alpha \mathbf{u}|^2 d\mathbf{x}\}. \quad (1.19)$$

Here ∂^α represents all the Eulerian mixed derivatives and we use \mathbf{u}, \mathbf{x} in order to emphasize they are vector quantities;

$$\begin{aligned} \mathcal{E}_L(t) \equiv & \frac{1}{2} \int_{x_0}^1 \chi u^2 dx + \frac{A}{\gamma-1} \int_{x_0}^1 \chi \rho^{\gamma-1} dx + \frac{1}{2} \sum_{j=1}^3 \left\{ \int_{x_0}^1 \chi |D_t^j u|^2 dx \right\} \\ & + \sum_{j=0}^2 \left\{ \frac{\mu}{2} \int_{x_0}^1 \chi \{ \rho r^4 |D_t^j D_x u|^2 + \frac{2|D_t^j u|^2}{\rho r^2} \} dx + \int_{x_0}^1 \chi \rho^{2\gamma-2} r^4 |D_t^j D_x \rho|^2 dx \right\} \\ & + \sum_{j=0}^1 \frac{1}{2} \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_t^j D_x^2 \rho|^2 dx + \frac{1}{2} \int_{x_0}^1 \chi \rho^{8\gamma-2} r^{12} |D_x^3 \rho|^2 dx. \end{aligned} \quad (1.20)$$

We first observe that some derivative terms are missing in $\mathcal{E}_L(t)$ and will show that missing terms are controlled by $\mathcal{E}_L(t)$ via the equations. More specifically, for the u -part, the terms involving more than one spatial derivative can be estimated by the momentum equation since they are closely related to

the viscosity term, which is represented by the sum of lower derivative terms. For the ρ -part, pure time derivative terms can be directly estimated by the continuity equation. We also observe that the energy defined in the above involves the pressure ($p = A\rho^\gamma$) rather than ρ itself; for instance, the first derivative energy of the ρ -part $\int_{x_0}^1 \chi \rho^{2\gamma-2} r^4 |D_x \rho|^2 dx$ is equal to $\frac{1}{(A\gamma)^2} \int_{x_0}^1 \chi r^4 |D_x p|^2 dx$. Indeed, the pressure turns out to be the right quantity to look at, and moreover, because the behavior of p ($\sim 1-x$) around the vacuum boundary in Lagrangian coordinates does not depend on γ , our analysis can embrace all the physically interesting γ 's. In particular, the different weights in front of different spatial derivatives of ρ have been carefully chosen so that not only the energy space can capture the behavior of stationary profiles, but also the energy estimates can be closed.

In turn we define the dissipation $\mathcal{D}(t) \equiv \mathcal{D}_E(t) + \mathcal{D}_L(t)$ by

$$\mu \sum_{|\alpha| \leq 3} \int \zeta |\nabla \partial^\alpha \mathbf{u}|^2 d\mathbf{x} + \mu \sum_{j=0}^3 \int_{x_0}^1 \chi \{ \rho r^4 |D_t^j D_x u|^2 + \frac{2|D_t^j u|^2}{\rho r^2} \} dx + \sum_{j=1}^3 \int_{x_0}^1 \chi |D_t^j u|^2 dx. \quad (1.21)$$

Next, we introduce the following assumption (K):

$$\begin{aligned} \sup_{x_0 \leq x \leq 1} \{ \rho, \left| \frac{u}{r} \right|, |\rho r^2 D_x u + \frac{2u}{r}| = \left| \frac{D_t \rho}{\rho} \right|, |\rho r^2 D_x u|, |\rho r^2 D_t D_x u|, \left| \frac{D_t u}{r} \right|, |\rho^{2\gamma-1} r^2 D_x \rho| \} &\leq K \\ \sup_{0 \leq r \leq r_2-d} \{ \rho, |u|, |\partial_r u|, \left| \frac{\partial_t \rho}{\rho} \right|, \left| \frac{\partial_r \rho}{\rho} \right|, |\partial_t u| \} &\leq K \end{aligned} \quad (1.22)$$

which indicates what regularity strong solutions should enjoy. It is shown in Lemma 4.1 that K is bounded by the energy \mathcal{E} . We are now ready to state the main a priori estimates.

Theorem 1.1. *Suppose ρ, u are smooth solutions to the free boundary problem of the Navier-Stokes-Poisson system (1.2) with (1.5), (1.6), and (1.7), or (1.12) with (1.13) for given initial data such that $\mathcal{E}(0) \equiv \mathcal{E}(\rho_{in}, u_{in})$ is bounded. Then there exist $C_1 = C_1(K)$, $C_2 > 0$ such that the following energy inequality holds for $0 \leq t < \frac{d}{K}$,*

$$\frac{d}{dt} \mathcal{E}(t) + \frac{1}{2} \mathcal{D}(t) \leq C_1 \mathcal{E}(t) + C_2 (\mathcal{E}(t))^2 \quad (1.23)$$

Moreover, there exist $T > 0$ and $A = A(T, C_1, C_2, \mathcal{E}(0)) > 0$ such that

$$\sup_{0 \leq t \leq T} \mathcal{E}(t) \leq A.$$

In the next theorem, we establish the local in time well-posedness of strong solutions to the Navier-Stokes-Poisson system.

Theorem 1.2. *Let the initial data ρ_{in}, u_{in} be given such that $\mathcal{E}(0) \equiv \mathcal{E}(\rho_{in}, u_{in}) < \infty$. There exists $T^* > 0$ such that there exists a unique solution $R(t)$, $\rho(t, r)$, $u(t, r)$ to the Navier-Stokes-Poisson system (1.2) with (1.5), (1.6), (1.7) in $[0, T^*) \times [0, R(t)]$ such that*

$$\sup_{0 \leq t \leq T^*} \mathcal{E}(t) \leq 2\mathcal{E}(0).$$

Moreover, $\rho(t, x)$, $u(t, x)$, $r(t, x)$ where x is a Lagrangian variable defined in (1.8), serve a unique solution to (1.12) with (1.13) in $[0, T^*) \times [0, 1]$.

We remark that the energy \mathcal{E} of the stationary solutions ρ_0 is bounded for all γ , and therefore strong solutions constructed in Theorem 1.2 can capture the physical vacuum boundary behavior (1.4) or (1.14) locally in time. We believe that this local well-posedness result provides the foundation

towards further interesting study such as global well-posedness under the same energy space and nonlinear stability questions of Lane-Emden steady stars.

The central difficulty in this article is to deal with the vacuum free boundary where the density vanishes at certain rate, which makes the system degenerate along the boundary. Since the free boundary gets fixed in Lagrangian coordinates, it is desirable to work in Lagrangian framework and the degeneracy from the vacuum is overcome with the density-weighted energy estimates. We note that these energy estimates can be closed due to the presence of the viscosity; the smoothing effect of the velocity is transferred to the density. We should also point out that the vacuum boundary condition $\rho(t, R(t)) = 0$ solely cannot define the free boundary properly but rather the dynamic boundary condition (1.7) characterizes the free boundary problem: indeed, it enables to integrate by parts up to the boundary. On the other hand, the three-dimensional structure of symmetric flows is still prevalent around the origin, the coordinate singularity from symmetry is worrisome, and thus the cooperation of Eulerian formulation seems necessary. We perform Lagrangian and Eulerian energy estimates separately by using the cutoff functions. Overlapping terms involving χ' and $\zeta'(\nabla\zeta)$ do not cause any other essential difficulty and are controlled by the opposite energies.

Another key idea is to implement an appropriate iteration scheme whose approximate solutions converge to the desired strong solution to the system under the energy norm \mathcal{E} . This can be done by the separation of the density and the velocity from the system in Lagrangian coordinates. With given fixed ρ^n , r^n related coefficients, we obtain u^{n+1} by solving linear parabolic PDEs. ρ^{n+1} is defined from the continuity equation along the particle path by using ρ^n , r^n , and u^{n+1} . In turn, r^{n+1} is determined by ρ^{n+1} from the dynamics of r . The $(n+1)$ -th total energy \mathcal{E}^{n+1} is accordingly further separated into ρ -part and u -part and we prove that they are uniformly bounded. We remark that this whole energy separation works due to the viscosity term, in other words, the viscosity dominates the given vacuum problem.

The method developed in this paper is lucid and concrete, and moreover, it provides the critical quantities, such as $\rho r^2 D_x u + \frac{2u}{r}$, which govern the whole dynamics. We believe that this method itself will have rich applications to other interesting problems. We note that our results strongly depend on the given initial data in that the initial density is explicitly embedded in the construction of the cutoff functions, which are important components of the energy \mathcal{E} .

It would be very interesting, challenging both physically and mathematically to study the full system without the symmetry assumption as a free boundary problem; in the general case, no result is known for compressible gas flows with the free boundary. We note that the existence problem for the pure compressible Navier-Stokes system in three space dimension is still open. Currently, the above argument does not seem to extend directly to the general case. However, we believe that the methodology can make a contribution to the study of the full system with the vacuum boundary, along with the recent progress on the free boundary problems from other contexts. We will leave them for future study.

The article proceeds as follows. In the first half of the paper, we establish the a priori estimates, which should shed some light on the construction of strong solutions. In Section 2, boundary estimates are performed in Lagrangian coordinates and in Section 3, interior estimates are presented in Eulerian coordinates. By weaving those two estimates and verifying that the smoothness assumption can be closed under the energy \mathcal{E} , we conclude the a priori estimates in Section 4. In the rest of the article, strong solutions are constructed by implementing an iteration scheme, based on the separation of the density and the velocity from the system. In Section 5, the approximate scheme is displayed and approximate solutions at each step are shown to have the same regularity of the previous approximations. Section 6 is devoted to obtaining uniform energy estimates and Theorem 1.2 is proven.

2 Boundary Estimates in Lagrangian Coordinates

First, we observe that the behavior of the density $\rho(t, x)$ in Lagrangian coordinates should be inherited from the initial profile ρ_{in} . This can be readily seen from the continuity equation in (1.12):

$$\rho(t, x) = \rho_{in}(x) \exp\left\{-\int_0^t (\rho r^2 D_x u + \frac{2u}{r})(\tau, x) d\tau\right\} \quad (2.1)$$

As long as $|\rho r^2 D_x u + \frac{2u}{r}|$ is bounded, so is ρ ; in Section 4, we will show that

$$\sup_{0 < x < 1} |\rho r^2 D_x u + \frac{2u}{r}| \leq C_{in}(\mathcal{E}(t)^{\frac{1}{2}} + \mathcal{E}(t))$$

where C_{in} depends only on the initial density ρ_{in} . Indeed, the quantity $|\rho r^2 D_x u + \frac{2u}{r}|$ as well as the formula ρ (2.1) are essential to establish the local well-posedness. Let

$$M \equiv \sup_{0 < x < 1} |\rho r^2 D_x u + \frac{2u}{r}|.$$

Note that $M \leq K$, where K appears in (1.22). We also observe that $\rho(t, x)$ can be controlled by ρ_{in} and M from (2.1): for $0 \leq t \leq T$

$$\rho_{in}(x)e^{-MT} \leq \rho(t, x) \leq \rho_{in}(x)e^{MT} \quad (2.2)$$

Thus for sufficiently small T , ρ stays close to initial density ρ_{in} . Integrating the momentum equation in x , one might hope to get $\mu(\rho r^2 D_x u + \frac{2u}{r}) = p + \int_1^x (\frac{D_t u}{r^2} + \frac{4\pi y}{r^4}) dy$. However, we remark that this computation is absurd because u does not have to be zero on $x = 1$, while $(\mu \rho r^2 D_x u - p)(t, 1) = 0$.

We establish the estimates of $\mathcal{E}_L(t)$ by a series of lemmas. The following lemma treats t -derivatives of u .

Lemma 2.1. *There exists $C_K > 0$ such that*

$$\frac{1}{2} \frac{d}{dt} \sum_{i=0}^3 \int_{x_0}^1 \chi |D_t^i u|^2 dx + \frac{3\mu}{4} \sum_{i=0}^3 \int_{x_0}^1 \chi \{ \rho r^4 |D_t^i D_x u|^2 + \frac{2|D_t^i u|^2}{\rho r^2} \} dx \leq C_K \mathcal{E}_L + \mathcal{O}\mathcal{L}_1, \quad (2.3)$$

where $\mathcal{O}\mathcal{L}_1 \leq \tilde{C}_K \mathcal{E}_E$ for some constant \tilde{C}_K .

Proof. It is instructive to see how the estimates go in detail at the energy level. Let $i = 0$. Multiply the momentum equation in (1.12) by χu and integrate to get

$$\frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi u^2 dx + \int_{x_0}^1 \chi u r^2 D_x p dx - \mu \int_{x_0}^1 \chi u D_x (\rho r^4 D_x u) dx + \int_{x_0}^1 \chi u \frac{4\pi x}{r^2} dx + \mu \int_{x_0}^1 \chi \frac{2u^2}{\rho r^2} dx = 0. \quad (2.4)$$

The second and third terms in (2.4): by integrating by parts and using the boundary condition (1.13)

$$\begin{aligned} \int_{x_0}^1 \chi u r^2 D_x p dx - \mu \int_{x_0}^1 \chi u D_x (\rho r^4 D_x u) dx &= - \int_{x_0}^1 D_x (\chi u r^2) p dx + \mu \int_{x_0}^1 D_x (\chi u) \rho r^4 D_x u dx \\ &= - \int_{x_0}^1 \chi' u r^2 p dx + \mu \int_{x_0}^1 \chi' u \rho r^4 D_x u dx - \underbrace{\int_{x_0}^1 \chi D_x (u r^2) p dx}_{\star} + \mu \int_{x_0}^1 \chi \rho r^4 |D_x u|^2 dx \end{aligned}$$

Use the continuity equation to reduce \star to

$$\star = \frac{A}{\gamma - 1} \frac{d}{dt} \int_{x_0}^1 \chi \rho^{\gamma-1} dx.$$

Overlapping terms can be controlled as follows:

$$\begin{aligned}
& - \int_{x_0}^{x_1} \chi' u r^2 p dx + \mu \int_{x_0}^{x_1} \chi' u \rho r^4 D_x u dx \leq \frac{C}{x_1 - x_0} \left\{ \int_{r_0(t)}^{r_1(t)} \rho^{\gamma+1} |u| r^4 dr + \mu \int_{r_0(t)}^{r_1(t)} \rho |u \partial_r u| r^4 dr \right\} \\
& \leq \frac{C(r_1 + d)^2}{x_1 - x_0} \left\{ \sup_{r \leq r_1 + d} \rho^{\gamma+1} \left(\int_{r_0(t)}^{r_1(t)} \rho^\gamma r^2 dr + \int_{r_0(t)}^{r_1(t)} \rho u^2 r^2 dr \right) + \int_{r_0(t)}^{r_1(t)} \rho u^2 r^2 dr + \int_{r_0(t)}^{r_1(t)} \rho |\partial_r u|^2 r^2 dr \right\} \\
& \leq \tilde{C}_K \mathcal{E}_E
\end{aligned}$$

where we have used the relations $dx = \rho r^2 dr$; $D_x = \frac{1}{\rho r^2} \partial_r$ as well as the Cauchy-Schwarz inequality. As for the fourth term in (2.4), we apply the Cauchy-Schwarz inequality:

$$\begin{aligned}
\int_{x_0}^1 \chi u \frac{4\pi x}{r^2} dx & \leq \frac{\mu}{2} \int_{x_0}^1 \chi \frac{u^2}{\rho r^2} dx + \frac{8\pi^2}{\mu} \int_{x_0}^1 \chi \rho \frac{x^2}{r^2} dx \\
& \leq \frac{\mu}{2} \int_{x_0}^1 \chi \frac{u^2}{\rho r^2} dx + \frac{C}{(r_0 - d)^2} \sup_{x_0 \leq x \leq 1} |\rho^{2-\gamma}| \int_{x_0}^1 \chi \rho^{\gamma-1} dx
\end{aligned}$$

where we have used $\rho = \rho^{2-\gamma} \rho^{\gamma-1}$ at the last inequality. After absorbing the viscosity term into the LHS and using the assumption (1.22), we get the following:

$$\begin{aligned}
& \frac{d}{dt} \int_{x_0}^1 \chi \left\{ \frac{1}{2} u^2 + \frac{A}{\gamma-1} \rho^{\gamma-1} \right\} dx + \frac{3\mu}{4} \int_{x_0}^1 \chi \{ \rho r^4 |D_x u|^2 + \frac{2u^2}{\rho r^2} \} dx \\
& \leq C_K \int_{x_0}^1 \chi \rho^{\gamma-1} dx + \tilde{C}_K \mathcal{E}_E
\end{aligned}$$

Note that from the continuity equation, we also get

$$\int_{x_0}^1 \chi \rho^{-3} |D_t \rho|^2 dx \leq 3 \int_{x_0}^1 \chi \{ \rho r^4 |D_x u|^2 + \frac{2u^2}{\rho r^2} \} dx. \quad (2.5)$$

Now let $i = 1$. Take D_t of the momentum equation to get

$$\begin{aligned}
& D_t^2 u + r^2 D_x D_t p - \mu D_x (\rho r^4 D_x D_t u) + \mu \frac{2D_t u}{\rho r^2} \\
& = \mu D_x (D_t (\rho r^4) D_x u) - D_t r^2 D_x p - D_t \left(\frac{2\mu}{\rho r^2} \right) u - D_t \left(\frac{4\pi x}{r^2} \right)
\end{aligned}$$

Multiply by $\chi D_t u$ and integrate in x to get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi |D_t u|^2 dx + \underbrace{\int_{x_0}^1 \chi D_t u r^2 D_x D_t p dx - \mu \int_{x_0}^1 \chi D_t u D_x (\rho r^4 D_t D_x u) dx}_{(i)} + \mu \int_{x_0}^1 \chi \frac{2|D_t u|^2}{\rho r^2} dx \\
& = \underbrace{\int_{x_0}^1 \chi \{ \mu D_x (D_t (\rho r^4) D_x u) - D_t r^2 D_x p - D_t \left(\frac{2\mu}{\rho r^2} \right) u - D_t \left(\frac{4\pi x}{r^2} \right) \} D_t u dx}_{(ii)}
\end{aligned}$$

The LHS can be estimated in the same way as in the zeroth order case. The RHS has new terms but it consists of lower order derivative terms. We will provide the detailed computation in the below. First, integrate by parts by using the boundary condition (1.13) in (i) to have

$$\begin{aligned}
(i) & = -A\gamma \underbrace{\int_{x_0}^1 \chi' D_t u r^2 \rho^{\gamma-1} D_t \rho dx}_{(a)} + \mu \int_{x_0}^1 \chi' D_t u \rho r^4 D_x D_t u dx \\
& \quad - A\gamma \underbrace{\int_{x_0}^1 \chi (r^2 D_t D_x u + \frac{2D_t u}{\rho r}) \rho^{\gamma-1} D_t \rho dx}_{(b)} + \mu \int_{x_0}^1 \chi \rho r^4 |D_t D_x u|^2 dx
\end{aligned}$$

Note that $D_x D_t u$ in (a) is not worrisome: by another integration by parts, (a) becomes

$$(a) = - \int_{x_0}^{x_1} \chi'' \rho r^4 |D_t u|^2 dx - \int_{x_0}^{x_1} \chi' D_x(\rho r^4) |D_t u|^2 dx,$$

which is bounded by the Eulerian energy. For (b), we apply the Cauchy-Schwarz inequality:

$$\begin{aligned} (b) &\leq \frac{\mu}{8} \int_{x_0}^1 \chi \{ \rho r^4 |D_t D_x u|^2 + \frac{|D_t u|^2}{\rho r^2} \} dx + \frac{10A^2 \gamma^2}{\mu} \int_{x_0}^1 \chi \rho^{2\gamma-3} |D_t \rho|^2 dx \\ &\leq \frac{\mu}{4} \int_{x_0}^1 \chi \{ \rho r^4 |D_t D_x u|^2 + \frac{|D_t u|^2}{\rho r^2} \} dx + \frac{30A^2 \gamma^2}{\mu} \sup_{x_0 \leq x \leq 1} |\rho^{2\gamma}| \int_{x_0}^1 \chi \{ \rho r^4 |D_x u|^2 + \frac{2u^2}{\rho r^2} \} dx \end{aligned} \quad (2.6)$$

where we have used (2.5). Each term in the LHS (ii) is treated as follows: For the first term we integrate by parts using the boundary condition.

$$\mu \int_{x_0}^1 \chi D_x(D_t(\rho r^4) D_x u) D_t u dx = -\mu \int_{x_0}^1 \chi' D_t(\rho r^4) D_x u D_t u dx - \underbrace{\mu \int_{x_0}^1 \chi D_t(\rho r^4) D_x u D_x D_t u dx}_{(c)}$$

The first term in the RHS is an overlapping term, bounded by $\tilde{C}_K \mathcal{E}_E$, and we apply the Cauchy-Schwarz inequality for the second term.

$$(c) \leq \frac{\mu}{8} \int_{x_0}^1 \chi \rho r^4 |D_t D_x u|^2 dx + 2\mu \sup_{x_0 \leq x \leq 1} \left| \frac{D_t \rho}{\rho} + \frac{4u}{r} \right|^2 \int_{x_0}^1 \chi \rho r^4 |D_x u|^2 dx$$

By recalling $D_t r = u$ and applying the Cauchy-Schwarz inequality, the second term in (ii) is bounded by $CK\mathcal{E}_L$ as follows:

$$\begin{aligned} - \int_{x_0}^1 \chi D_t r^2 D_x p D_t u dx &= -2A \int_{x_0}^1 \chi \frac{u}{r} r^2 \rho^{\gamma-1} D_x \rho D_t u dx \\ &\leq A \sup_{x_0 \leq x \leq 1} \left| \frac{u}{r} \right| \left\{ \int_{x_0}^1 \chi \rho^{2\gamma-2} r^4 |D_x \rho|^2 dx + \int_{x_0}^1 \chi |D_t u|^2 dx \right\} \end{aligned}$$

In the same way, one can check that the third term in (ii) is bounded by

$$-2\mu \int_{x_0}^1 \chi D_t \left(\frac{1}{\rho r^2} \right) u D_t u dx \leq \frac{\mu}{8} \int_{x_0}^1 \chi \frac{|D_t u|^2}{\rho r^2} dx + 8\mu \sup_{x_0 \leq x \leq 1} \left| \frac{D_t \rho}{\rho} + \frac{2u}{r} \right|^2 \int_{x_0}^1 \chi \frac{u^2}{\rho r^2} dx$$

The last term in (ii) is estimated in the following

$$-4\pi \int_{x_0}^1 \chi D_t \left(\frac{x}{r^2} \right) D_t u dx = 8\pi \int_{x_0}^1 \chi \frac{ux}{r^3} D_t u dx \leq \frac{\mu}{8} \int_{x_0}^1 \chi \frac{|D_t u|^2}{\rho r^2} dx + \frac{32\pi^2}{\mu} \sup_{x_0 \leq x \leq 1} \left| \frac{\rho}{r^4} \right| \int_{x_0}^1 \chi u^2 dx$$

Thus, after absorbing the viscosity term into the LHS and using the assumption (1.22), we get (2.3) for $i = 1$. Now let $i = 2$ or 3 . The spirit is same as in the case $i = 1$. Take D_t^i of the momentum equation to get

$$\begin{aligned} D_t^{i+1} u + r^2 D_x D_t^i p - \mu D_x(\rho r^4 D_x D_t^i u) + \mu \frac{2D_t^i u}{\rho r^2} \\ = \sum_{j=0}^{i-1} \{ \mu D_x(D_t^{i-j}(\rho r^4) D_t^j D_x u) - D_t^{i-j} r^2 D_x D_t^j p - D_t^{i-j} \left(\frac{2\mu}{\rho r^2} \right) D_t^j u \} - D_t^i \left(\frac{4\pi x}{r^2} \right) \end{aligned} \quad (2.7)$$

Multiply by $\chi D_t^i u$ and integrate in x to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi |D_t^i u|^2 dx + \int_{x_0}^1 \chi D_t^i u r^2 D_x D_t^i p dx - \mu \int_{x_0}^1 \chi D_t^i u D_x (\rho r^4 D_x D_t^i u) dx + \mu \int_{x_0}^1 \chi \frac{2|D_t^i u|^2}{\rho r^2} dx \\ &= \sum_{j=0}^{i-1} \int_{x_0}^1 \chi \{ \mu D_x (D_t^{i-j} (\rho r^4) D_t^j D_x u) - D_t^{i-j} r^2 D_x D_t^j p - D_t^{i-j} (\frac{2\mu}{\rho r^2}) D_t^j u \} D_t^i u dx \\ & \quad - \int_{x_0}^1 \chi D_t^i u D_t^i (\frac{4\pi x}{r^2}) dx \end{aligned}$$

Note each term in the RHS involves only lower order derivatives. The second and third terms in the LHS can be written as the following: by integrating by parts and using the boundary condition (1.13)

$$- \int_{x_0}^1 \chi' D_t^i u r^2 D_t^i p dx + \mu \int_{x_0}^1 \chi' D_t^i u \rho r^4 D_x D_t^i u dx - \underbrace{\int_{x_0}^1 \chi D_x (r^2 D_t^i u) D_t^i p dx}_{\star} + \mu \int_{x_0}^1 \chi \rho r^4 |D_x D_t^i u|^2 dx$$

The first two terms are overlapping terms and they can be bounded by $\tilde{C}_K \mathcal{E}_E$ by using the change of variable $D_t = \partial_t + u \partial_r$. One might try to use the continuity equation to reduce \star to a t -derivative as in the zeroth order case, but we point out that it is rather complicated. Instead, we use the Cauchy-Schwarz inequality and take advantage of the viscosity term to control it as we did in (2.6). In order to do so, it is enough to derive the following: for $0 \leq i \leq 2$,

$$\begin{aligned} D_t^{i+1} \rho &= -\rho \{ \rho r^2 D_x D_t^i u + \frac{2D_t^i u}{r} \} - \sum_{0 \leq j < i, 0 \leq k \leq i} D_t^{i-j-k} \rho^2 D_x (D_t^k r^2 D_t^j u) \\ &\Rightarrow \int_{x_0}^1 \chi \rho^{-3} |D_t^j \rho|^2 dx \leq C_K \mathcal{E}_L \text{ for } 1 \leq j \leq 3. \end{aligned} \tag{2.8}$$

Hence, by summing over i , we get the following:

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^3 \int_{x_0}^1 \chi |D_t^i u|^2 dx + \frac{3\mu}{4} \sum_{i=1}^3 \int_{x_0}^1 \chi \{ \rho r^4 |D_x D_t^i u|^2 + \frac{2|D_t^i u|^2}{\rho r^2} \} dx \leq C_K \mathcal{E}_L + \tilde{C}_K \mathcal{E}_E.$$

This finishes the proof of Lemma 2.1. \square

Next, we estimate mixed derivatives with only one spatial derivative.

Lemma 2.2. *There exists $C_K > 0$ such that*

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \sum_{i=0}^2 \int_{x_0}^1 \chi \{ \rho r^4 |D_x D_t^i u|^2 + \frac{2|D_t^i u|^2}{\rho r^2} \} dx + \frac{1}{2} \sum_{i=0}^2 \int_{x_0}^1 \chi |D_t^{i+1} u|^2 dx \\ & \leq \frac{\mu}{4} \sum_{i=1}^3 \int_{x_0}^1 \chi \{ \rho r^4 |D_x D_t^i u|^2 + \frac{2|D_t^i u|^2}{\rho r^2} \} dx + C_K \mathcal{E}_L + \mathcal{O}\mathcal{L}_2 \end{aligned} \tag{2.9}$$

where $\mathcal{O}\mathcal{L}_2 \leq \tilde{C}_K \mathcal{E}_E$.

Note that part of the dissipation energy, which involves one more derivative terms than the total energy, appears in the RHS of the inequality (2.9). This is not a problem because it will be absorbed into the dissipation obtained in Lemma 2.1.

Proof. We provide the estimate for $i = 0$ in detail. Higher order terms can be estimated in the same way, since taking t derivative does not destroy the structure of the equations. Multiply the momentum equation by $D_t u$ and integrate to get:

$$\begin{aligned} \int_{x_0}^1 \chi |D_t u|^2 dx + \int_{x_0}^1 \chi (r^2 D_x p) D_t u dx - \mu \int_{x_0}^1 \chi D_x (\rho r^4 D_x u) D_t u dx + \mu \int_{x_0}^1 \chi \frac{2u D_t u}{\rho r^2} dx \\ + \int_{x_0}^1 \chi \frac{4\pi x}{r^2} D_t u dx = 0 \end{aligned}$$

The second term: by the Cauchy-Schwarz inequality

$$\begin{aligned} \left| \int_{x_0}^1 \chi (r^2 D_x p) D_t u dx \right| &\leq \int_{x_0}^1 \chi r^4 |D_x p|^2 dx + \frac{1}{4} \int_{x_0}^1 \chi |D_t u|^2 dx \\ &= (A\gamma)^2 \int_{x_0}^1 \chi \rho^{2\gamma-2} |D_x \rho|^2 dx + \frac{1}{4} \int_{x_0}^1 \chi |D_t u|^2 dx \end{aligned}$$

The third term: by integrating by parts and using the boundary condition (1.13)

$$\begin{aligned} -\mu \int_{x_0}^1 \chi D_x (\rho r^4 D_x u) D_t u dx &= \mu \int_{x_0}^{x_1} \chi' \rho r^4 D_x u D_t u dx + \mu \int_{x_0}^1 \chi \rho r^4 D_x u D_x D_t u dx \\ &\quad \star \\ \star &= \frac{\mu}{2} \frac{d}{dt} \int_{x_0}^1 \chi \rho r^4 |D_x u|^2 dx - \frac{\mu}{2} \int_{x_0}^1 \chi D_t \rho r^4 |D_x u|^2 dx - 2\mu \int_{x_0}^1 \chi \rho r^3 u |D_x u|^2 dx \end{aligned}$$

Recall $\sup_{x_0 \leq x \leq 1} \left| \frac{D_t \rho}{\rho} \right| \leq K$ and $\sup_{x_0 \leq x \leq 1} \left| \frac{u}{r} \right| \leq K$ from the assumption (1.22). Thus we get

$$\star \leq \frac{\mu}{2} \frac{d}{dt} \int_{x_0}^1 \chi \rho r^4 |D_x u|^2 dx + \frac{5\mu K}{2} \int_{x_0}^1 \chi \rho r^4 |D_x u|^2 dx.$$

The fourth term: first apply the product rule in t -variable and use the assumption (1.22)

$$\begin{aligned} \mu \int_{x_0}^1 \chi \frac{2u D_t u}{\rho r^2} dx &= \frac{\mu}{2} \frac{d}{dt} \int_{x_0}^1 \chi \frac{2u^2}{\rho r^2} dx + \mu \int_{x_0}^1 \chi \frac{D_t \rho u^2}{\rho^2 r^2} dx + 2\mu \int_{x_0}^1 \chi \frac{u^3}{\rho r^3} dx \\ &\leq \frac{\mu}{2} \frac{d}{dt} \int_{x_0}^1 \chi \frac{2u^2}{\rho r^2} dx + \frac{3\mu K}{2} \int_{x_0}^1 \chi \frac{2u^2}{\rho r^2} dx \end{aligned}$$

The fifth term: apply the Cauchy-Schwarz inequality

$$\begin{aligned} \int_{x_0}^1 \chi \frac{4\pi x}{r^2} D_t u dx &\leq \frac{\mu}{4} \int_{x_0}^1 \chi \frac{|D_t u|^2}{\rho r^2} dx + \frac{1}{\mu} \int_{x_0}^1 \chi \frac{16\pi^2 x^2 \rho}{r^2} dx \\ &\leq \frac{\mu}{4} \int_{x_0}^1 \chi \frac{|D_t u|^2}{\rho r^2} dx + \frac{C}{(r_0 - d)^2} \sup_{x_0 \leq x \leq 1} |\rho^{2-\gamma}| \int_{x_0}^1 \chi \rho^{\gamma-1} dx \end{aligned}$$

This completes the inequality (2.9) for $i = 0$. For $i = 1, 2$, multiply (2.7) by $\chi D_t^{i+1} u$ and integrate to get

$$\begin{aligned} \int_{x_0}^1 \chi |D_t^{i+1} u|^2 dx + \int_{x_0}^1 \chi r^2 D_x D_t^i p D_t u^{i+1} dx - \mu \int_{x_0}^1 \chi D_x (\rho r^4 D_x D_t^i u) D_t u^{i+1} dx \\ + \mu \int_{x_0}^1 \chi \frac{2D_t^i u D_t u^{i+1}}{\rho r^2} dx = - \int_{x_0}^1 \chi D_t^i \left(\frac{4\pi x}{r^2} \right) D_t^{i+1} u dx \\ + \sum_{j=0}^{i-1} \int_{x_0}^1 \chi \{ \mu D_x (D_t^{i-j} (\rho r^4) D_t^j D_x u) - D_t^{i-j} r^2 D_x D_t^j p - D_t^{i-j} \left(\frac{2\mu}{\rho r^2} \right) D_t^j u \} D_t^{i+1} u dx \end{aligned}$$

The LHS has the same structure as the case $i = 0$ and the RHS consists of lower order terms. The computation is similar as in the previous lemma. Hence, by applying the Cauchy-Schwarz inequality and using the assumption (K), we get the desired results. \square

Before we move onto higher spatial derivatives of u , we present how to estimate $D_x \rho$, a crucial step to close the energy estimates. Note that for stationary solutions, due to (1.14),

$$|D_x \rho_0^{\frac{\gamma}{2}}|^2 \sim \frac{1}{1-x}$$

which is not integrable, and hence weighted estimates come into play. In order to estimate spatial derivatives of ρ , we make use of both the continuity equation and an integral form of the momentum equation. We point out that the estimate does not yield overlapping terms.

Lemma 2.3. *There exists $C_K > 0$ such that*

$$\frac{1}{2} \frac{d}{dt} \sum_{i=0}^2 \int_{x_0}^1 \chi \rho^{2\gamma-2} r^4 |D_t^i D_x \rho|^2 dx \leq C_K \mathcal{E}_L. \quad (2.10)$$

Proof. Integrate the momentum equation in x from 1 to x for $x \geq x_0$: due to the boundary condition (1.13),

$$-\rho r^2 D_x u = -\frac{1}{\mu} p - \frac{1}{\mu} \int_1^x \left\{ \frac{D_t u}{r^2} + \frac{4\pi y}{r^4} + \mu \frac{2u}{\rho r^4} - \mu \frac{2D_x u}{r} \right\} dy. \quad (2.11)$$

We will estimate ρ^γ rather than ρ . Multiply by $\gamma \rho^\gamma$ and use the continuity equation:

$$\gamma \rho^\gamma \frac{D_t \rho}{\rho} = -\frac{\gamma}{\mu} \rho^\gamma p - \gamma \rho^\gamma \frac{2u}{r} - \frac{\gamma}{\mu} \rho^\gamma \int_1^x \left\{ \frac{D_t u}{r^2} + \frac{4\pi y}{r^4} + \mu \frac{2u}{\rho r^4} - \mu \frac{2D_x u}{r} \right\} dy$$

Apply D_x :

$$\begin{aligned} D_t D_x(\rho^\gamma) &= -\frac{2A\gamma}{\mu} \rho^\gamma D_x(\rho^\gamma) - \gamma D_x(\rho^\gamma) \frac{2u}{r} - \gamma \rho^\gamma D_x\left(\frac{2u}{r}\right) - \frac{\gamma}{\mu} \rho^\gamma \left(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} + \mu \frac{2u}{\rho r^4} - \mu \frac{2D_x u}{r} \right) \\ &\quad - \frac{\gamma}{\mu} D_x(\rho^\gamma) \int_1^x \left\{ \frac{D_t u}{r^2} + \frac{4\pi y}{r^4} + \mu \frac{2u}{\rho r^4} - \mu \frac{2D_x u}{r} \right\} dy \end{aligned}$$

Use the identity $D_x(\frac{u}{r}) = \frac{D_x u}{r} - \frac{u}{\rho r^4}$ and (2.11) in order to simplify the above as the following:

$$D_t D_x(\rho^\gamma) = -\frac{A\gamma}{\mu} \rho^\gamma D_x(\rho^\gamma) - \gamma D_x(\rho^\gamma) \left(\rho r^2 D_x u + \frac{2u}{r} \right) - \frac{\gamma}{\mu} \rho^\gamma \left(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} \right)$$

Multiply by $r^4 D_x(\rho^\gamma)$, and integrate to get:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi r^4 |D_x(\rho^\gamma)|^2 dx - 2 \int_{x_0}^1 \chi r^3 u |D_x(\rho^\gamma)|^2 dx + \frac{A\gamma}{\mu} \int_{x_0}^1 \chi \rho^\gamma r^4 |D_x(\rho^\gamma)|^2 dx \\ &= -\gamma \int_{x_0}^1 \chi \left(\rho r^2 D_x u + \frac{2u}{r} \right) r^4 |D_x(\rho^\gamma)|^2 dx - \frac{\gamma}{\mu} \int_{x_0}^1 \chi \rho^\gamma \left(D_t u + \frac{4\pi x}{r^2} \right) r^2 D_x(\rho^\gamma) dx \end{aligned} \quad (2.12)$$

The last term can be bounded by

$$\frac{\gamma}{\mu} \int_{x_0}^1 \chi \rho^\gamma \left(D_t u + \frac{4\pi x}{r^2} \right) r^2 D_x(\rho^\gamma) dx \leq \frac{A\gamma}{2\mu} \int_{x_0}^1 \chi \rho^\gamma r^4 |D_x(\rho^\gamma)|^2 dx + \frac{1}{A\mu} \int_{x_0}^1 \chi \rho^\gamma \left(|D_t u|^2 + \frac{16\pi^2 x^2}{r^4} \right) dx$$

by the Cauchy-Schwarz inequality. Hence by the assumption (1.22), (2.12) becomes

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi r^4 |D_x(\rho^\gamma)|^2 dx + \frac{A\gamma}{2\mu} \int_{x_0}^1 \chi \rho^\gamma r^4 |D_x(\rho^\gamma)|^2 dx \\ & \leq (2 + \gamma) K \int_{x_0}^1 \chi r^4 |D_x(\rho^\gamma)|^2 dx + C \sup_{x_0 \leq x \leq 1} |\rho^\gamma| \int_{x_0}^1 \chi |D_t u|^2 dx + C \sup_{x_0 \leq x \leq 1} |\rho| \int_{x_0}^1 \chi \rho^{\gamma-1} dx \end{aligned}$$

This proves (2.10) when $i = 0$. Note that taking t derivatives does not destroy the structure of equations and thus, by using the previous lemmas for t derivatives of u , one can derive the similar estimates for $i = 1, 2$. \square

As a corollary of Lemma 2.3, we derive the estimates of mixed derivatives of u with two spatial derivatives, which will be needed to estimate higher order derivatives of ρ and to close the energy estimates later in Section 4, directly from the equation. The L^2 estimate of $D_x(\rho r^2 D_x u + \frac{2u}{r})$ easily follows from the momentum equation: square it and integrate to get

$$\begin{aligned} \int_{x_0}^1 \chi \rho |r^2 D_x(\rho r^2 D_x u + \frac{2u}{r})|^2 dx & \leq \frac{2}{\mu^2} \int_{x_0}^1 \chi \{ \rho |D_t u|^2 + \rho r^4 |D_x p|^2 + \frac{16\pi^2 x^2 \rho}{r^4} \} dx \\ & \leq C_K \mathcal{E}_L \end{aligned} \quad (2.13)$$

Each term in the RHS has been already estimated and it is bounded by \mathcal{E}_L . Next, in order to estimate $D_x(\rho r^2 D_x u)$, we rewrite the momentum equation:

$$\mu D_x(\rho r^2 D_x u) = \frac{D_t u}{r^2} + D_x p + \frac{4\pi x}{r^4} + \mu \frac{2u}{\rho r^4} - \mu \frac{2D_x u}{r}$$

Note that due to the singularity of the last two terms at the vacuum boundary in RHS we need an appropriate weight for L^2 estimate of $D_x(\rho r^2 D_x u)$: in the same spirit as in (2.13), we obtain

$$\begin{aligned} \frac{\mu^2}{2} \int_{x_0}^1 \chi \rho |r^2 D_x(\rho r^2 D_x u)|^2 dx & \leq \int_{x_0}^1 \chi \{ \rho |D_t u|^2 + \rho r^4 |D_x p|^2 + \frac{16\pi^2 x^2 \rho}{r^4} \\ & \quad + \frac{4\mu^2 u^2}{\rho r^4} + 4\mu^2 \rho r^2 |D_x u|^2 \} dx \\ & \leq C_K \mathcal{E}_L + \frac{4\mu^2}{(r_0 - d)^2} \int_{x_0}^1 \chi \{ \rho r^4 |D_x u|^2 + \frac{2u^2}{\rho r^2} \} dx \end{aligned} \quad (2.14)$$

Similarly, one can derive the following: for $i = 1, 2$

$$\begin{aligned} \int_{x_0}^1 \chi \rho |r^2 D_x(\rho r^2 D_t^i D_x u + \frac{2D_t^i u}{r})|^2 dx & \leq C_K \mathcal{E}_L \\ \int_{x_0}^1 \chi \rho |r^2 D_x(\rho r^2 D_t^i D_x u)|^2 dx & \leq (C_K + C_{r_0}) \mathcal{E}_L \end{aligned} \quad (2.15)$$

where C_{r_0} is a constant depending on r_0 . But this dependence on r_0 can be ignored because we have chosen r_0 and d so that $\frac{1}{r_0 - d} \leq 1$.

Next, we present the weighted energy estimate of $D_x^2 \rho$.

Lemma 2.4. *There exists $C_K > 0$ such that*

$$\frac{1}{2} \frac{d}{dt} \sum_{i=0}^1 \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_t^i D_x^2 \rho|^2 dx \leq C_K \mathcal{E}_L. \quad (2.16)$$

Proof. Take D_x of the continuity equation and use the momentum equation to substitute $D_x(\rho^2 D_x u + \frac{2u}{r})$ by $\frac{1}{\mu}(\frac{D_t u}{r^2} + A\gamma\rho^{\gamma-1}D_x\rho + \frac{4\pi x}{r^4})$:

$$D_t D_x \rho + D_x \rho (\rho^2 D_x u + \frac{2u}{r}) + \frac{\rho}{\mu} (\frac{D_t u}{r^2} + A\gamma\rho^{\gamma-1}D_x\rho + \frac{4\pi x}{r^4}) = 0 \quad (2.17)$$

Take one more D_x of (2.17), use the momentum equation again to substitute $D_x(\rho^2 D_x u + \frac{2u}{r})$ by $\frac{1}{\mu}(\frac{D_t u}{r^2} + A\gamma\rho^{\gamma-1}D_x\rho + \frac{4\pi x}{r^4})$, and rearrange terms to get:

$$\begin{aligned} D_t D_x^2 \rho + \frac{A\gamma}{\mu} \rho^\gamma D_x^2 \rho &= -D_x^2 \rho (\rho^2 D_x u + \frac{2u}{r}) - \frac{A\gamma(\gamma+1)}{\mu} \rho^{\gamma-1} |D_x \rho|^2 \\ &\quad - \frac{2D_x \rho}{\mu} (\frac{D_t u}{r^2} + \frac{4\pi x}{r^4}) - \frac{\rho}{\mu} D_x (\frac{D_t u}{r^2} + \frac{4\pi x}{r^4}) \end{aligned} \quad (2.18)$$

Multiply by $\chi \rho^{4\gamma-2} r^8 D_x^2 \rho$ and integrate in x to get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx - \frac{1}{2} \int_{x_0}^1 \chi D_t (\rho^{4\gamma-2} r^8) |D_x^2 \rho|^2 dx + \frac{A\gamma}{\mu} \int_{x_0}^1 \chi \rho^{5\gamma-2} |D_x^2 \rho|^2 dx \\ &= - \int_{x_0}^1 \chi (\rho^2 D_x u + \frac{2u}{r}) \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx - \frac{A\gamma(\gamma+1)}{\mu} \int_{x_0}^1 \chi \rho^{5\gamma-3} r^8 |D_x \rho|^2 D_x^2 \rho dx \\ &\quad - \frac{2}{\mu} \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 D_x \rho (\frac{D_t u}{r^2} + \frac{4\pi x}{r^4}) D_x^2 \rho dx \\ &\quad - \frac{1}{\mu} \int_{x_0}^1 \chi \rho^{4\gamma-1} r^8 (\frac{D_t D_x u}{r^2} - \frac{2D_t u}{\rho r^5} + \frac{4\pi}{r^4} - \frac{16\pi x}{\rho r^7}) D_x^2 \rho dx \end{aligned}$$

(i) (ii) (iii) (iv) (v)

For (i) and (ii), we use the assumption (1.22):

$$\begin{aligned} (i) &= (2\gamma-1) \int_{x_0}^1 \chi D_t \rho \rho^{4\gamma-3} r^8 |D_x^2 \rho|^2 dx + 4 \int_{x_0}^1 \chi \rho^{4\gamma-2} r^7 u |D_x^2 \rho|^2 dx \\ &\leq (2\gamma-1) \sup_{x_0 \leq x \leq 1} \left| \frac{D_t \rho}{\rho} \right| \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx + 4 \sup_{x_0 \leq x \leq 1} \left| \frac{u}{r} \right| \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \\ &\leq CK \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \\ (ii) &\leq \sup_{x_0 \leq x \leq 1} \left| \rho^2 D_x u + \frac{2u}{r} \right| \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \leq K \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \end{aligned}$$

For (iii) and (iv), we use the assumption (1.22) and apply the Cauchy-Schwarz inequality:

$$\begin{aligned} (iii) &\leq C \sup_{x_0 \leq x \leq 1} |\rho^{2\gamma-1} r^2 D_x \rho| \left(\int_{x_0}^1 \chi \rho^{2\gamma-2} r^4 |D_x \rho|^2 dx + \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \right) \\ (iv) &\leq C \sup_{x_0 \leq x \leq 1} |\rho^{2\gamma-1} r^2 D_x \rho| \left(\int_{x_0}^1 \chi (|D_t u|^2 + \frac{16\pi^2 x^2}{r^4}) dx + \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \right) \end{aligned}$$

For (v), we separate terms into two cases. Apply the Cauchy-Schwarz inequality to the first two terms

after taking the sup of $\rho^{2\gamma-\frac{1}{2}}$:

$$\begin{aligned} & \int_{x_0}^1 \chi \rho^{4\gamma-1} r^8 \left(\frac{D_t D_x u}{r^2} - \frac{2D_t u}{\rho r^5} \right) D_x^2 \rho dx \\ & \leq \sup_{x_0 \leq x \leq 1} |\rho^{2\gamma-\frac{1}{2}}| \left(\int_{x_0}^1 \chi (\rho r^4 |D_t D_x u|^2 + \frac{2|D_t u|^2}{\rho r^2}) dx + \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \right) \end{aligned}$$

For the last two terms, before applying the Cauchy-Schwarz inequality, we take the sup of different powers of ρ in order to have them bounded by \mathcal{E}_L :

$$\begin{aligned} 4\pi \int_{x_0}^1 \chi \rho^{4\gamma-1} r^4 D_x^2 \rho dx & \leq 2\pi \sup_{x_0 \leq x \leq 1} |\rho^{\frac{3\gamma+1}{2}}| \left(\int_{x_0}^1 \chi \rho^{\gamma-1} dx + \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \right) \\ -16\pi \int_{x_0}^1 \chi \rho^{4\gamma-2} r x D_x^2 \rho dx & \leq 8\pi \sup_{x_0 \leq x \leq 1} |\rho^{\frac{3\gamma-1}{2}}| \left(\frac{1}{(r_0-d)^6} \int_{x_0}^1 \chi \rho^{\gamma-1} dx + \int_{x_0}^1 \chi \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \right) \end{aligned}$$

This completes (2.16) for $i = 0$. One can derive the similar inequality when $i = 1$. \square

As a direct result of the previous lemmas, we can get the bounds of mixed derivatives of u with three spatial derivatives. Multiply the momentum equation by ρr^2 and take D_x to get

$$\begin{aligned} \mu D_x(\rho r^2 D_x(\rho r^2 D_x u)) & = \rho D_t D_x u + D_x \rho D_t u + A\gamma \rho^\gamma r^2 D_x^2 \rho + A\gamma^2 \rho^{\gamma-1} r^2 |D_x \rho|^2 + \frac{2A\gamma \rho^{\gamma-1} D_x \rho}{r} \\ & \quad + \frac{4\pi \rho}{r^2} + \frac{4\pi x D_x \rho}{r^2} - \frac{8\pi x}{r^5} + \frac{4\mu D_x u}{r^2} - \frac{4\mu u}{\rho r^5} - \frac{2\mu D_x(\rho r^2 D_x u)}{r}. \end{aligned}$$

Each term in RHS has been estimated with suitable weights, and thus one can check

$$\mu^2 \int_{x_0}^1 \chi |\rho r^2 D_x(\rho r^2 D_x(\rho r^2 D_x u))|^2 dx \leq C_K \mathcal{E}_L. \quad (2.19)$$

Similarly, one gets

$$\mu^2 \int_{x_0}^1 \chi |\rho r^2 D_x(\rho r^2 D_x(\rho r^2 D_t D_x u))|^2 dx \leq C_K \mathcal{E}_L.$$

To complete the estimate for \mathcal{E}_L , it remains to estimate pure spatial derivative terms $D_x^3 \rho$ and $D_x^4 u$. The spirit is same as in Lemma 2.4: take D_x of (2.18) get

$$\begin{aligned} D_t D_x^3 \rho + \frac{A\gamma}{\mu} \rho^\gamma D_x^3 \rho & = -D_x^3 \rho (\rho r^2 D_x u + \frac{2u}{r}) - \frac{3A\gamma(\gamma+1)}{\mu} \rho^{\gamma-1} D_x \rho D_x^2 \rho - \frac{A\gamma(\gamma^2-1)}{\mu} \rho^{\gamma-2} (D_x \rho)^3 \quad (i) \\ & \quad - \frac{3D_x^2 \rho}{\mu} \left(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} \right) - \frac{3D_x \rho}{\mu} D_x \left(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} \right) - \frac{\rho}{\mu} D_x^2 \left(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} \right) \quad (ii) \end{aligned}$$

Each term in RHS has been already treated with appropriate weights and thus by multiplying by $\rho^{8\gamma-2} r^{12} D_x^3 \rho$ and integrating, we conclude the following:

$$\frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi \rho^{8\gamma-2} r^{12} |D_x^3 \rho|^2 dx \leq C_K \mathcal{E}_L \quad (2.20)$$

Note that the weight $\rho^{8\gamma-2}$ is carefully chosen such that the RHS can be bounded by $C_K \mathcal{E}_L$. For instance, (i), (ii) can be estimated as follows. Indeed, the estimate of (i) will provide a good reason for the choice of weights. First, for (i), we take the sup of $\rho^{2\gamma-1} r^2 D_x \rho$ and apply the Cauchy-Schwarz

inequality.

$$\begin{aligned} \int_{x_0}^1 \chi \rho^{8\gamma-2} r^{12} \rho^{\gamma-2} (D_x \rho)^3 D_x^3 \rho dx &= \int_{x_0}^1 \chi (\rho^{2\gamma-1} r^2 D_x \rho)^2 (\rho^{\gamma-1} r^2 D_x \rho) (\rho^{4\gamma-1} r^6 D_x^3 \rho) dx \\ &\leq \sup_{x_0 \leq x \leq 1} |\rho^{2\gamma-1} r^2 D_x \rho|^2 \left(\int_{x_0}^1 \chi \rho^{2\gamma-2} r^4 |D_x \rho|^2 dx + \int_{x_0}^1 \chi \rho^{8\gamma-2} r^{12} |D_x^3 \rho|^2 dx \right) \leq K^2 \mathcal{E}_L \end{aligned}$$

For (ii), first compute the second derivative of $\frac{D_t u}{r^2} + \frac{4\pi x}{r^4}$.

$$\begin{aligned} \int_{x_0}^1 \chi \rho^{8\gamma-2} r^{12} \rho D_x^2 \left(\frac{D_t u}{r^2} + \frac{4\pi x}{r^4} \right) D_x^3 \rho dx &= \int_{x_0}^1 \chi \rho^{4\gamma-1} \{ r^2 D_x (\rho r^2 D_t D_x u) - (\rho r^2 D_t D_x u) \left(\frac{D_x \rho r^2}{\rho} + \frac{6}{\rho r} \right) \right. \\ &\quad \left. + 2 D_t u \left(\frac{D_x \rho r}{\rho} + \frac{5}{\rho r^2} \right) - \frac{32\pi}{r} + 16\pi x \left(\frac{D_x \rho}{\rho r} + \frac{7}{\rho r^4} \right) \} (\rho^{4\gamma-1} r^6 D_x^3 \rho) dx \end{aligned}$$

Each term in the RHS has been already estimated. We repeat the same argument, namely, extract an appropriate factor of ρ possibly with some other terms outside the integral to make use of the assumption (K), and apply the Cauchy-Schwarz inequality to conclude that it is bounded by $C_K \mathcal{E}_L$ for some constant C_K depending on K . In turn, from the momentum equation, a weighted L^2 estimate of $D_x(\rho r^2 D_x(\rho r^2 D_x(\rho r^2 D_x u)))$ can be obtained. This completes the estimate of $\mathcal{E}_L(t)$.

3 Interior Estimates in Eulerian Coordinates

In this section, we present the interior estimates of $\mathcal{E}_E(t)$. Away from the vacuum boundary, ρ is expected to be strictly positive and the classical results of the Navier-Stokes theory can be applied. The potential terms need attention. Interior domain containing the origin retains three-dimensional structure despite the symmetry and H^3 interior estimate is necessary. The Eulerian description is used to establish the interior energy estimates. The energy estimates will be performed in the cartesian product space in order to avoid the coordinate singularity at the origin coming from the symmetry. Recall the full Navier-Stokes-Poisson system (1.1):

$$\begin{aligned} \rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0 \\ \rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}) + \nabla p + \rho \nabla \Phi &= \mu \Delta \mathbf{u} \\ \Delta \Phi &= 4\pi \rho \end{aligned}$$

where $\mathbf{u}(\mathbf{x}) = u(r) \frac{\mathbf{x}}{r}$, where $\mathbf{x} = (x_1, x_2, x_3)$, $r = |\mathbf{x}|$.

We will not put the potential term into conservation form in the energy estimates because it makes the energy negative definite and in principle, it serves lower order term and it can be controlled by main terms ρ, u . The symmetrizer of the system is used for the energy estimates to get cleaner estimates, while other weight functions may work by the aid of the dissipation.

Lemma 3.1. *There exist constants $C_K, C > 0$ such that*

$$\frac{1}{2} \frac{d}{dt} \mathcal{E}_E + \frac{1}{2} \mathcal{D}_E \leq C_K \mathcal{E}_E + C(\mathcal{E}_E)^2 + \mathcal{O}\mathcal{L}_3 \quad (3.1)$$

where $\mathcal{O}\mathcal{L}_3 \leq \tilde{C}_K \mathcal{E}_L$ for some \tilde{C}_K .

Proof. It is standard to perform the energy estimates to the Navier-Stokes equations, and so we point out the gist of the estimates rather than provide all the details. First, the zeroth order energy estimate is carried out. Let $W_0 \equiv \frac{A\gamma}{\gamma-1} \rho^{\gamma-1}$ be a symmetrizing weight function. Multiply (1.1) by ζW_0 and $\zeta \mathbf{u}$,

and integrate to get:

$$\begin{aligned}
& \frac{d}{dt} \left\{ \frac{1}{2} \int \zeta \rho |\mathbf{u}|^2 d\mathbf{x} + \frac{A}{\gamma-1} \int \zeta \rho^\gamma d\mathbf{x} \right\} + \mu \int \zeta |\nabla \mathbf{u}|^2 d\mathbf{x} \\
&= \frac{1}{4\pi} \int \zeta \nabla \Phi \cdot \nabla \partial_t \Phi d\mathbf{x} + \frac{1}{2} \int \zeta \partial_t \rho |\mathbf{u}|^2 d\mathbf{x} - \int \zeta \rho (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{u} d\mathbf{x} \\
&+ \frac{A\gamma}{\gamma-1} \int \nabla \zeta \cdot \rho^\gamma \mathbf{u} d\mathbf{x} + \mu \int \nabla \zeta \cdot \nabla \mathbf{u} \mathbf{u} d\mathbf{x} + \int \nabla \zeta \Phi \rho \mathbf{u} d\mathbf{x} + \frac{1}{4\pi} \int \nabla \zeta \Phi \nabla \Phi_t d\mathbf{x}
\end{aligned}$$

The RHS consists of the first potential term, the next two nonlinear terms, and remaining overlapping terms. Here is the estimate of the potential term: by symmetry

$$\begin{aligned}
\left| \frac{1}{4\pi} \int \zeta \nabla \Phi \cdot \nabla \partial_t \Phi d\mathbf{x} \right| &= \left| \int_0^{r_2-d} \zeta \left(\frac{1}{r^3} \int_0^r \rho s^2 ds \right) \left(\frac{1}{r^3} \int_0^r \partial_t \rho s^2 ds \right) r^2 dr \right| \\
&= \left| - \int_0^{r_2-d} \zeta \left(\frac{1}{r^3} \int_0^r \rho s^2 ds \right) \left(\frac{\rho u}{r} \right) r^2 dr \right| \\
&\leq \frac{1}{\mu} \int_0^{r_2-d} \zeta \left(\frac{1}{r^3} \int_0^r \rho s^2 ds \right)^2 \rho^2 r^2 dr + \frac{\mu}{4} \int \zeta \left| \frac{u}{r} \right|^2 r^2 dr \\
&\leq \frac{1}{9\mu} \sup_{0 \leq r \leq r_2-d} |\rho^{4-\gamma}| \int \zeta \rho^\gamma r^2 dr + \frac{\mu}{4} \int \zeta \left| \frac{u}{r} \right|^2 r^2 dr
\end{aligned}$$

We have applied the Cauchy-Schwarz inequality and have used for $r \leq r_2 - d$,

$$\frac{1}{r^3} \int_0^r \rho s^2 ds \leq \frac{1}{3} \sup_{0 \leq r \leq r_2-d} |\rho|.$$

Next, we compute overlapping terms in Lagrangian coordinates. To illustrate the idea, we present the estimate of the first two terms: by change of variables, Eulerian integrals are converted into Lagrangian integrals and then we apply the Cauchy-Schwarz inequality.

$$\begin{aligned}
\left| \int \nabla \zeta \cdot \rho^\gamma \mathbf{u} d\mathbf{x} \right| &\leq \frac{C}{r_2 - r_1 - 2d} \int_{x_1}^{x_2} \rho^{\gamma-1} |u| dx \\
&\leq \frac{C}{r_2 - r_1 - 2d} \sup_{x_1 \leq x \leq x_2} |\rho^{\frac{\gamma-1}{2}}| \left\{ \int_{x_1}^{x_2} \rho^{\gamma-1} dx + \int_{x_1}^{x_2} |u|^2 dx \right\} \\
\left| \mu \int \nabla \zeta \cdot \nabla \mathbf{u} \mathbf{u} d\mathbf{x} \right| &= \left| - \frac{\mu}{2} \int \Delta \zeta |\mathbf{u}|^2 d\mathbf{x} \right| \\
&\leq C\mu \int_{x_1}^{x_2} u^2 \frac{dx}{\rho} \leq C\mu \sup_{x_1 \leq x \leq x_2} \left| \frac{1}{\rho} \right| \int_{x_1}^{x_2} |u|^2 dx
\end{aligned}$$

Note that

$$\sup_{x_1 \leq x \leq x_2} \left| \frac{1}{\rho} \right| \leq \sup_{x_1 \leq x \leq x_2} \left| \frac{1}{\rho_{in}} \right| e^{MT} \text{ from (2.2).}$$

Hence we get the following zeroth-order estimate:

$$\frac{1}{2} \frac{d}{dt} \left\{ \int \zeta \rho |\mathbf{u}|^2 d\mathbf{x} + \frac{A}{\gamma-1} \int \zeta \rho^\gamma d\mathbf{x} \right\} + \frac{3\mu}{4} \int \zeta |\nabla \mathbf{u}|^2 d\mathbf{x} \leq C_K \mathcal{E}_E + \mathcal{O}\mathcal{L} \quad (3.2)$$

where $\mathcal{O}\mathcal{L} \leq C_{K,in} \mathcal{E}_L$ and $C_{K,in}$ depends also on the initial density ρ_{in} . The higher temporal derivatives (up to third) can be estimated in the similar way. Let ∂ be any Eulerian derivative.

$$\begin{aligned}
& \partial_t(\partial \rho) + \nabla \partial \rho \cdot \mathbf{u} + \nabla \rho \cdot \partial \mathbf{u} + \partial \rho \nabla \cdot \mathbf{u} + \rho \nabla \cdot \partial \mathbf{u} = 0 \\
& \rho \partial_t(\partial \mathbf{u}) + \partial \rho \partial_t \mathbf{u} + \partial[\rho(\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla \partial p + \rho \nabla \partial \Phi + \partial \rho \nabla \Phi = \mu \Delta \partial \mathbf{u}
\end{aligned}$$

Define $W_1 \equiv A\gamma\rho^{\gamma-2}\partial\rho$.

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \{ A\gamma \int \zeta \rho^{\gamma-2} (\partial\rho)^2 d\mathbf{x} + \int \zeta \rho |\partial\mathbf{u}|^2 d\mathbf{x} \} + \mu \int \zeta |\nabla \partial\mathbf{u}|^2 d\mathbf{x} + \mu \int \nabla \zeta \nabla \partial\mathbf{u} \partial\mathbf{u} d\mathbf{x} \\
&= \frac{A\gamma(\gamma-2)}{2} \int \zeta \rho^{\gamma-3} \partial_t \rho (\partial\rho)^2 d\mathbf{x} + \frac{1}{2} \int \zeta \partial_t \rho |\partial\mathbf{u}|^2 d\mathbf{x} - A\gamma \int \zeta \rho^{\gamma-2} \partial\rho \nabla \partial\rho \cdot \mathbf{u} d\mathbf{x} \\
&\quad - A\gamma \int \zeta \rho^{\gamma-2} \partial\rho \nabla \rho \cdot \partial\mathbf{u} d\mathbf{x} - A\gamma \int \zeta \rho^{\gamma-2} (\partial\rho)^2 \nabla \cdot \mathbf{u} d\mathbf{x} - A\gamma \int \zeta \rho^{\gamma-1} \partial\rho \nabla \cdot \partial\mathbf{u} d\mathbf{x} \\
&\quad - \int \zeta \partial\rho \partial_t \mathbf{u} \partial\mathbf{u} d\mathbf{x} - \int \zeta \partial[\rho(\mathbf{u} \cdot \nabla) \mathbf{u}] \partial\mathbf{u} d\mathbf{x} - \int \zeta \nabla \partial p \partial\mathbf{u} d\mathbf{x} - \int \zeta \rho \nabla \partial\Phi \partial\mathbf{u} d\mathbf{x} - \int \zeta \partial\rho \nabla \Phi \partial\mathbf{u} d\mathbf{x}
\end{aligned}$$

Note that the pressure term $-\int \zeta \nabla \partial p \partial\mathbf{u} d\mathbf{x}$ on the RHS contains the undesirable second derivative of ρ but it is canceled with $-A\gamma \int \zeta \rho^{\gamma-1} \partial\rho \nabla \cdot \partial\mathbf{u} d\mathbf{x}$ due to the choice of symmetrizing weight W_1 :

$$-A\gamma \int \zeta \rho^{\gamma-1} \partial\rho \nabla \cdot \partial\mathbf{u} d\mathbf{x} - \int \zeta \nabla \partial p \partial\mathbf{u} d\mathbf{x} = - \int \zeta \partial p \nabla \cdot \partial\mathbf{u} d\mathbf{x} - \int \zeta \nabla \partial p \partial\mathbf{u} d\mathbf{x} = \int \nabla \zeta \partial p \cdot \partial\mathbf{u} d\mathbf{x}$$

For another second derivative term $-A\gamma \int \zeta \rho^{\gamma-2} \partial\rho \nabla \partial\rho \cdot \mathbf{u} d\mathbf{x}$, we integrate it by parts:

$$\frac{A\gamma}{2} \int \nabla \zeta \rho^{\gamma-2} (\partial\rho)^2 \mathbf{u} d\mathbf{x} + \frac{A\gamma(\gamma-2)}{2} \int \zeta \rho^{\gamma-3} \nabla \rho (\partial\rho)^2 \mathbf{u} d\mathbf{x} + \frac{A\gamma}{2} \int \zeta \rho^{\gamma-2} (\partial\rho)^2 \nabla \cdot \mathbf{u} d\mathbf{x}$$

Potential terms are, in principle, lower order and the L^2 estimate $\|\partial^2 \Phi\|_{L^2} \leq C\|\rho\|_{L^2}$ is useful. Hence we get the following first order estimate:

$$\frac{1}{2} \frac{d}{dt} \{ A\gamma \int \zeta \rho^{\gamma-2} (\partial\rho)^2 d\mathbf{x} + \int \zeta \rho |\partial\mathbf{u}|^2 d\mathbf{x} \} + \frac{3\mu}{4} \int \zeta |\nabla \partial\mathbf{u}|^2 d\mathbf{x} \leq C_K \mathcal{E}_E + \mathcal{O}\mathcal{L} \quad (3.3)$$

In order to apply the Sobolev imbedding theorem, we need to estimate up to the 3rd derivatives. Take one more derivative of the equations:

$$\begin{aligned}
& \partial_t (\partial^2 \rho) + \nabla \partial^2 \rho \cdot \mathbf{u} + 2 \nabla \partial \rho \cdot \partial \mathbf{u} + \nabla \rho \cdot \partial^2 \mathbf{u} + \partial^2 \rho \nabla \cdot \mathbf{u} + 2 \partial \rho \nabla \cdot \partial \mathbf{u} + \rho \nabla \cdot \partial^2 \mathbf{u} = 0 \\
& \rho \partial_t (\partial^2 \mathbf{u}) + 2 \partial \rho \partial_t (\partial \mathbf{u}) + \partial^2 \rho \partial_t \mathbf{u} + \partial^2 [\rho(\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla \partial^2 p + 2 \partial \rho \nabla \partial \Phi + \rho \nabla \partial^2 \Phi + \partial^2 \rho \nabla \Phi = \mu \Delta \partial^2 \mathbf{u}
\end{aligned}$$

Let $W_2 \equiv A\gamma\rho^{\gamma-2}\partial^2\rho$.

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \{ \int \zeta \rho^{\gamma-2} (\partial^2 \rho)^2 d\mathbf{x} + \int \zeta \rho |\partial^2 \mathbf{u}|^2 d\mathbf{x} \} + \mu \int \zeta |\nabla \partial^2 \mathbf{u}|^2 d\mathbf{x} + \mu \int \nabla \zeta \nabla \partial^2 \mathbf{u} \partial^2 \mathbf{u} d\mathbf{x} \\
&= \frac{A\gamma(\gamma-2)}{2} \int \zeta \rho^{\gamma-3} \partial_t \rho (\partial^2 \rho)^2 d\mathbf{x} + \frac{1}{2} \int \zeta \partial_t \rho |\partial^2 \mathbf{u}|^2 d\mathbf{x} - A\gamma \int \zeta \rho^{\gamma-2} \partial^2 \rho \nabla \partial^2 \rho \cdot \mathbf{u} d\mathbf{x} \\
&\quad - 2A\gamma \int \zeta \rho^{\gamma-2} \partial^2 \rho \nabla \partial \rho \cdot \partial \mathbf{u} d\mathbf{x} - A\gamma \int \zeta \rho^{\gamma-2} \partial^2 \rho \nabla \rho \cdot \partial^2 \mathbf{u} d\mathbf{x} - A\gamma \int \zeta \rho^{\gamma-2} (\partial^2 \rho)^2 \nabla \cdot \mathbf{u} d\mathbf{x} \\
&\quad - 2A\gamma \int \zeta \rho^{\gamma-2} \partial^2 \rho \partial \rho \nabla \cdot \partial \mathbf{u} d\mathbf{x} - A\gamma \int \zeta \rho^{\gamma-1} \partial^2 \rho \nabla \cdot \partial^2 \mathbf{u} d\mathbf{x} - 2 \int \zeta \partial \rho \partial_t (\partial \mathbf{u}) \partial^2 \mathbf{u} d\mathbf{x} \\
&\quad - \int \zeta \partial^2 \rho \partial_t \mathbf{u} \partial^2 \mathbf{u} d\mathbf{x} - \int \zeta \partial^2 [\rho(\mathbf{u} \cdot \nabla) \mathbf{u}] \partial^2 \mathbf{u} d\mathbf{x} - \int \zeta \nabla \partial^2 p \cdot \partial^2 \mathbf{u} d\mathbf{x} - 2 \int \zeta \partial \rho \nabla \partial \Phi \cdot \partial^2 \mathbf{u} d\mathbf{x} \\
&\quad - \int \zeta \rho \nabla \partial^2 \Phi \cdot \partial^2 \mathbf{u} d\mathbf{x} - \int \zeta \partial^2 \rho \nabla \Phi \cdot \partial^2 \mathbf{u} d\mathbf{x}
\end{aligned}$$

As in the first order estimates, for higher order derivative terms, either we use the integration by parts or they cancel each other. Eventually, we get the following:

$$\frac{1}{2} \frac{d}{dt} \{ A\gamma \int \zeta \rho^{\gamma-2} (\partial^2 \rho)^2 d\mathbf{x} + \int \zeta \rho |\partial^2 \mathbf{u}|^2 d\mathbf{x} \} + \frac{3\mu}{4} \int \zeta |\nabla \partial^2 \mathbf{u}|^2 d\mathbf{x} \leq C_K \mathcal{E}_E + \mathcal{O}\mathcal{L} \quad (3.4)$$

Take one more derivative and perform the weighted energy estimates with $W_3 = A\gamma\rho^{\gamma-2}\partial^3\rho$. It is routine to have the following high energy inequality:

$$\frac{1}{2}\frac{d}{dt}\{A\gamma\int\zeta\rho^{\gamma-2}(\partial^3\rho)^2d\mathbf{x} + \int\zeta\rho|\partial^3\mathbf{u}|^2d\mathbf{x}\} + \frac{\mu}{2}\int\zeta|\nabla\partial^3\mathbf{u}|^2d\mathbf{x} \leq C_K\mathcal{E}_E + C(\mathcal{E}_E)^2 + \mathcal{O}\mathcal{L} \quad (3.5)$$

We remark that $C(\mathcal{E}_E)^2$ comes from the Gagliardo-Nirenberg inequality:

$$\|f\|_{L^4} \leq \frac{1}{2}\|\nabla f\|_{L^2}^{\frac{3}{4}}\|f\|_{L^2}^{\frac{1}{4}}$$

to treat the nonlinear terms such as $\int\partial^2\rho\partial_t\partial\mathbf{u}\partial^3\mathbf{u}d\mathbf{x}$. The application of the Gagliardo-Nirenberg inequality appears later in the article. The spirit is same, so we omit details. Note that a overlapping term $\mu\int\nabla\zeta\nabla\partial^3\mathbf{u}\partial^3\mathbf{u}d\mathbf{x}$, which has overfull derivative, can be also bounded by $C\mathcal{E}_L$ because it can be reduced to $-\frac{\mu}{2}\int\Delta\zeta|\partial^3\mathbf{u}|^2d\mathbf{x}$ after integration by parts; a ρ -weighted norm of $D_x(\rho r^2 D_x(\rho r^2 D_x u))$ can be controlled by $C\mathcal{E}_L$; ρ has a uniform upper bound as well as a uniform lower bound depending only on the initial profile in the overlapping interval. This finishes the proof of the lemma. \square

Proof. of Theorem 1.1: From the previous lemmas and the energy inequalities, (1.23) follows. It remains to show that \mathcal{E} is bounded. We estimate A explicitly by solving the differential inequality (1.23). By separation of variables,

$$\int\frac{d\mathcal{E}}{C_1\mathcal{E} + C_2(\mathcal{E})^2} \leq \int dt \Rightarrow \ln\left|\frac{\mathcal{E}(t)}{\frac{C_1}{C_2} + \mathcal{E}(t)}\right| \leq \ln\left|\frac{\mathcal{E}(0)}{\frac{C_1}{C_2} + \mathcal{E}(0)}\right| + C_1 t \Rightarrow \frac{\mathcal{E}(t)}{\frac{C_1}{C_2} + \mathcal{E}(t)} \leq \frac{\mathcal{E}(0)}{\frac{C_1}{C_2} + \mathcal{E}(0)} e^{C_1 t}$$

If $e^{C_1 t} < \frac{C_1 + C_2\mathcal{E}(0)}{C_2\mathcal{E}(0)}$, we can derive the following:

$$\mathcal{E}(t) \leq \frac{C_1\mathcal{E}(0)e^{C_1 t}}{C_1 - C_2\mathcal{E}(0)(e^{C_1 t} - 1)} \equiv A$$

Note that $T < \frac{1}{C_1} \ln\left|\frac{C_1 + C_2\mathcal{E}(0)}{C_2\mathcal{E}(0)}\right|$. This completes the proof of Theorem 1.1. \square

In the next section, we verify the assumption (K) to show that the energy estimates can be closed for $0 \leq t \leq T$ where T is sufficiently small.

4 Weaving the Estimates

The a priori estimates in the previous sections hold under the assumption (K): (1.22). In order to close the estimates, it remains to show that the assumption can be closed under the same energy space \mathcal{E} . It results from the application of the Sobolev embedding theorems for one dimension as well as three dimension by combining boundary and interior estimates. We remark that the positivity of ρ having a lower and upper bounds in the overlapping region is critical, which results from the formula (2.1).

Lemma 4.1. *Suppose that ρ, u, r serve a smooth solution to the Navier-Stokes-Poisson system. Then there exist $T > 0$, $C = C(\rho_{in}) > 0$ such that $K \leq C\{\mathcal{E}(t)^{\frac{1}{2}} + \mathcal{E}(t)\}$ for $0 \leq t \leq T$.*

Proof. We provide the detailed computation for a few terms in the assumption. Other terms can be verified in the same way. First of all, we pay attention to ρ . For clear presentation, we use M , instead of K , for the bound of $|\rho r^2 D_x u + \frac{2u}{r}|$: $|\rho r^2 D_x u + \frac{2u}{r}| \leq M$. In the view of (2.1), ρ can be controlled by M :

$$\rho_{in} e^{-MT} \leq \rho(t, x) \leq \rho_{in} e^{MT}$$

With these bounds of ρ , let us estimate $|\rho r^2 D_x u + \frac{2u}{r}|$.

$$\sup_{0 < x < 1} |\rho r^2 D_x u + \frac{2u}{r}| = \sup\left\{ \sup_{0 \leq r \leq r_1+d} |\partial_r u + \frac{2u}{r}|, \sup_{x_1 \leq x \leq 1} |\rho r^2 D_x u + \frac{2u}{r}| \right\}$$

Apply the Sobolev embedding theorem:

$$\begin{aligned} \bullet \sup_{0 \leq r \leq r_1+d} |\partial_r u + \frac{2u}{r}| &\leq \sum_{i=0}^2 \left(\int_{B_{r_1+d}} |\nabla \partial_{\mathbf{x}}^i \mathbf{u}|^2 d\mathbf{x} \right)^{\frac{1}{2}} \\ &\leq \sup_{0 \leq r \leq r_1+d} \left| \frac{1}{\sqrt{\rho}} \right| \sum_{i=0}^2 \left(\int_{B_{r_1+d}} \rho |\nabla \partial_{\mathbf{x}}^i \mathbf{u}|^2 d\mathbf{x} \right)^{\frac{1}{2}} \\ &\leq \sup_{0 \leq r \leq r_1+d} \left| \frac{1}{\sqrt{\rho_{in}}} \right| e^{\frac{MT}{2}} (\mathcal{E}_E(t))^{\frac{1}{2}} \end{aligned}$$

In addition to the Sobolev embedding theorem, apply the Hölder inequality:

$$\begin{aligned} \bullet \sup_{x_1 \leq x \leq 1} |\rho r^2 D_x u + \frac{2u}{r}| &\leq \int_{x_1}^1 |\rho r^2 D_x u + \frac{2u}{r}| dx + \int_{x_1}^1 |D_x(\rho r^2 D_x u + \frac{2u}{r})| dx \\ &\leq \left(\int_{x_1}^1 \rho dx \right)^{\frac{1}{2}} \left(\int_{x_1}^1 \{ \rho r^4 |D_x u|^2 + \frac{2u^2}{\rho r^2} \} dx \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{x_1}^1 \frac{1}{\rho r^4} dx \right)^{\frac{1}{2}} \left(\int_{x_1}^1 \rho |r^2 D_x(\rho r^2 D_x u + \frac{2u}{r})|^2 dx \right)^{\frac{1}{2}} \\ &\leq \sup_{x_1 \leq x \leq 1} |\rho_{in}^{\frac{2-\gamma}{2}}| e^{\frac{2-\gamma}{2} MT} \mathcal{E}_L(t) + \left(\frac{1}{r_1-d} - \frac{1}{R+d} \right)^{\frac{1}{2}} \cdot \\ &\quad \left\{ \sup_{x_1 \leq x \leq 1} |\sqrt{\rho_{in}}| e^{\frac{MT}{2}} + \frac{1}{(r_1-d)^2} \sup_{x_1 \leq x \leq 1} |\rho_{in}^{\frac{2-\gamma}{2}}| e^{\frac{2-\gamma}{2} MT} \right\} (\mathcal{E}_L(t))^{\frac{1}{2}} \end{aligned}$$

where we have used the fact

$$\int_{x_1}^1 \frac{1}{\rho r^4} dx = \int_{r_1(t)}^{R(t)} \frac{1}{r^2} dr \leq \frac{1}{r_1-d} - \frac{1}{R+d}.$$

Note that $\frac{1}{r_1-d} \leq \frac{1}{r_0-d} \leq 1$. Combining the above two inequalities, first we get

$$M \leq C_{in} e^{\frac{MT}{2}} \{ (\mathcal{E}(t))^{\frac{1}{2}} + \mathcal{E}(t) \}$$

where $C_{in} = \sup\{ \sup_{0 \leq r \leq r_1+d} |\frac{1}{\sqrt{\rho_{in}}}|, \sup_{x_1 \leq x \leq 1} |\sqrt{\rho_{in}}| \}$. By Taylor expansion,

$$M - C_{in} \{ (\mathcal{E}(t))^{\frac{1}{2}} + \mathcal{E}(t) \} \sum_{k=1}^{\infty} \frac{(MT)^k}{2^k k!} \leq C_{in} \{ (\mathcal{E}(t))^{\frac{1}{2}} + \mathcal{E}(t) \}$$

Note that for sufficiently small T

$$C_{in} \{ (\mathcal{E}(t))^{\frac{1}{2}} + \mathcal{E}(t) \} \sum_{k=1}^{\infty} \frac{(MT)^k}{2^k k!} \leq \frac{M}{2}$$

and therefore for such T , we get the following:

$$M \leq C_{in} \{ (\mathcal{E}(t))^{\frac{1}{2}} + \mathcal{E}(t) \} \text{ for } 0 \leq t \leq T. \quad (4.1)$$

We have the exact same bound for $|\frac{D_t \rho}{\rho}|$. Next, following the same path, we estimate $|\rho r^2 D_x u|$.

$$\begin{aligned}
\sup_{0 < x < 1} |\rho r^2 D_x u| &\leq \sup\left\{ \sup_{0 \leq r \leq r_1 + d} |\partial_r u|, \sup_{x_1 \leq x \leq 1} |\rho r^2 D_x u| \right\} \\
\bullet \sup_{0 \leq r \leq r_1 + d} |\partial_r u| &\leq \sum_{i=0}^2 \left(\int_{B_{r_1+d}} |\nabla \partial_{\mathbf{x}}^i \mathbf{u}|^2 d\mathbf{x} \right)^{\frac{1}{2}} \\
&\leq \sup_{0 \leq r \leq r_1 + d} \left| \frac{1}{\sqrt{\rho}} \right| \sum_{i=0}^2 \left(\int_{B_{r_1+d}} \rho |\nabla \partial_{\mathbf{x}}^i \mathbf{u}|^2 d\mathbf{x} \right)^{\frac{1}{2}} \\
&\leq \sup_{0 \leq r \leq r_1 + d} \left| \frac{1}{\sqrt{\rho_{in}}} \right| e^{\frac{MT}{2}} (\mathcal{E}_E(t))^{\frac{1}{2}} \\
\bullet \sup_{x_1 \leq x \leq 1} |\rho r^2 D_x u| &\leq \int_{x_1}^1 |\rho r^2 D_x u| dx + \int_{x_1}^1 |D_x(\rho r^2 D_x u)| dx \\
&\leq \left(\int_{x_1}^1 \rho dx \right)^{\frac{1}{2}} \left(\int_{x_1}^1 \rho r^4 |D_x u|^2 dx \right)^{\frac{1}{2}} + \left(\int_{x_1}^1 \frac{1}{\rho r^4} dx \right)^{\frac{1}{2}} \left(\int_{x_1}^1 \rho |r^2 D_x(\rho r^2 D_x u)|^2 dx \right)^{\frac{1}{2}} \\
&\leq \sup_{x_1 \leq x \leq 1} |\rho_{in}^{\frac{2-\gamma}{2}}| e^{\frac{2-\gamma}{2} MT} \mathcal{E}_L(t) + \left(\frac{1}{r_1 - d} - \frac{1}{R + d} \right)^{\frac{1}{2}} \\
&\quad \left\{ \sup_{x_1 \leq x \leq 1} |\sqrt{\rho_{in}}| e^{\frac{MT}{2}} + \frac{1}{(r_1 - d)^2} \sup_{x_1 \leq x \leq 1} |\rho_{in}^{\frac{2-\gamma}{2}}| e^{\frac{2-\gamma}{2} MT} + \frac{1}{(r_1 - d)^2} \right\} (\mathcal{E}_L(t))^{\frac{1}{2}}
\end{aligned}$$

Because of (4.1) and Taylor expansion, we can derive the following: for sufficiently small T and a constant C_{in} ,

$$\sup_{0 < x < 1} |\rho r^2 D_x u| \leq C_{in} \{ (\mathcal{E}(t))^{\frac{1}{2}} + \mathcal{E}(t) \} \text{ for } 0 \leq t \leq T.$$

Note that it also follows that $|\frac{u}{r}|$ is bounded in $0 < x < 1$. The argument works for $|\rho r^2 D_t D_x u|$ and $|\frac{D_t u}{r}|$ in the same way.

Next, we estimate $|\rho^{2\gamma-1} r^2 D_x \rho|$ in $x_0 \leq x \leq 1$. Because the cutoff function χ values 1 only for $x_1 \leq x \leq 1$, $|\rho^{2\gamma-1} r^2 D_x \rho|$ for $x_0 \leq x \leq x_1$ should be estimated in Eulerian coordinates. Note that $r_0 - d \leq r \leq r_1 + d$ covers $x_0 \leq x \leq 1$.

$$\sup_{x_0 \leq x \leq 1} |\rho^{2\gamma-1} r^2 D_x \rho| \leq \sup\left\{ \sup_{r_0-d \leq r \leq r_1+d} |\rho^{2\gamma-2} \partial_r \rho|, \sup_{x_1 \leq x \leq 1} |\rho^{2\gamma-1} r^2 D_x \rho| \right\}$$

Apply the L^1 Sobolev embedding theorem in one dimension and in turn the Hölder inequality.

$$\begin{aligned}
\bullet \sup_{r_0-d \leq r \leq r_1+d} |\rho^{2\gamma-2} \partial_r \rho| &\leq \int_{r_0-d}^{r_1+d} |\rho^{2\gamma-2} \partial_r \rho| dr + \int_{r_0-d}^{r_1+d} |\partial_r(\rho^{2\gamma-2} \partial_r \rho)| dr \\
&\leq \left(\int_{r_0-d}^{r_1+d} \frac{\rho^{3\gamma-2}}{r^2} dr \right)^{\frac{1}{2}} \sum_{i=1}^2 \left(\int_{r_0-d}^{r_1+d} \rho^{\gamma-2} |\partial_r^i \rho|^2 r^2 dr \right)^{\frac{1}{2}} + \int_{r_0-d}^{r_1+d} \rho^{2\gamma-3} |\partial_r \rho|^2 dr \\
&\leq \left\{ \frac{1}{r_0 - d} + 1 \right\} \sup_{r_0-d \leq r \leq r_1+d} |\rho_{in}^{\gamma-1}| e^{(\gamma-1)MT} \mathcal{E}_E(t)
\end{aligned}$$

$$\begin{aligned}
\bullet \sup_{x_1 \leq x \leq 1} |\rho^{2\gamma-1} r^2 D_x \rho| &\leq \int_{x_1}^1 |\rho^{2\gamma-1} r^2 D_x \rho| dx + \int_{x_1}^1 |D_x(\rho^{2\gamma-1} r^2 D_x \rho)| dx \\
&\leq \left(\int_{x_1}^1 \rho^{2\gamma} dx \right)^{\frac{1}{2}} \left(\int_{x_1}^1 \rho^{2\gamma-2} r^4 |D_x \rho|^2 dx \right)^{\frac{1}{2}} + \left(\int_{x_1}^1 \frac{1}{r^4} dx \right)^{\frac{1}{2}} \left(\int_{x_1}^1 \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \right)^{\frac{1}{2}} \\
&\quad + (2\gamma-1) \int_{x_1}^1 \rho^{2\gamma-2} r^2 |D_x \rho|^2 dx + \left(\int_{x_1}^1 \frac{\rho^{2\gamma-2}}{r^5} dx \right)^{\frac{1}{2}} \left(\int_{x_1}^1 \rho^{2\gamma-2} r^4 |D_x \rho|^2 dx \right)^{\frac{1}{2}} \\
&\leq \sup_{x_1 \leq x \leq 1} |\rho_{in}^{\frac{\gamma+1}{2}}| e^{\frac{\gamma+1}{2} MT} \mathcal{E}_L(t) + \frac{1}{(r_1-d)^2} \{(\mathcal{E}_L(t))^{\frac{1}{2}} + \mathcal{E}_L(t)\} \\
&\quad + \frac{1}{(r_1-d)^{\frac{5}{2}}} \sup_{x_1 \leq x \leq 1} |\rho_{in}^{\frac{\gamma-1}{2}}| e^{\frac{\gamma-1}{2} MT} \mathcal{E}_L(t)
\end{aligned}$$

Thus, as in the previous cases, we conclude that

$$\sup_{x_0 \leq x \leq 1} |\rho^{2\gamma-1} r^2 D_x \rho| \leq C_{in} \{(\mathcal{E}(t))^{\frac{1}{2}} + \mathcal{E}(t)\} \text{ for } 0 \leq t \leq T$$

for small enough T . And hence we have completed the verification of all the Lagrangian terms in (1.22). For Eulerian terms, we give the detail for $\sup_{0 \leq r \leq r_2-d} |\frac{\partial_r \rho}{\rho}|$. Other terms such as $\frac{\partial_t \rho}{\rho}$ and $\partial_t u$ can be estimated in the same way by using the change of variables: $\partial_t = D_t - r^2 \rho u D_x$ in the overlapping region to estimate them in Lagrangian interval $x_1 \leq x \leq x_2$. First we observe that it is enough to compute $\sup_{0 \leq r \leq r_2-d} |\partial_r \rho|$, since

$$\sup_{0 \leq r \leq r_2-d} \left| \frac{1}{\rho} \right| \leq \sup_{0 \leq r \leq r_2-d} \left| \frac{1}{\rho_{in}} \right| e^{MT}.$$

Here is the estimate of $\partial_r \rho$.

$$\begin{aligned}
\sup_{0 \leq r \leq r_2+d} |\partial_r \rho| &= \sup \left\{ \sup_{0 \leq r \leq r_1+d} |\partial_r \rho|, \sup_{r_1+d \leq r \leq r_2-d} |\partial_r \rho| \right\} \\
\bullet \sup_{0 \leq r \leq r_1+d} |\partial_r \rho| &\leq \sum_{|\alpha| \leq 2} \left(\int_{B_{r_1+d}} |\partial_{\mathbf{x}}^\alpha \partial_{\mathbf{x}} \rho|^2 d\mathbf{x} \right)^{\frac{1}{2}} \\
&\leq \sup_{0 \leq r \leq r_1+d} |\rho^{\frac{2-\gamma}{2}}| \left\{ \sum_{|\alpha| \leq 2} \left(\int_{B_{r_1+d}} \rho^{\gamma-2} |\partial_{\mathbf{x}}^\alpha \partial_{\mathbf{x}} \rho|^2 d\mathbf{x} \right)^{\frac{1}{2}} \right\} \\
&\leq \sup_{0 \leq r \leq r_1+d} |\rho_{in}^{\frac{2-\gamma}{2}}| e^{\frac{2-\gamma}{2} MT} (\mathcal{E}_E(t))^{\frac{1}{2}}
\end{aligned}$$

Note that $D_x(\rho r^2 D_x \rho) = \frac{2D_x \rho}{r} + r^2 |D_x \rho|^2 + \rho r^2 D_x^2 \rho$.

$$\begin{aligned}
\bullet \sup_{r_1+d \leq r \leq r_2-d} |\partial_r \rho| &\leq \sup_{x_1 \leq x \leq x_2} |\rho r^2 D_x \rho| \\
&\leq \int_{x_1}^{x_2} |\rho r^2 D_x \rho| dx + \int_{x_1}^{x_2} |D_x(\rho r^2 D_x \rho)| dx \\
&\leq \left(\int_{x_1}^{x_2} \left\{ \rho^{4-2\gamma} + \frac{1}{\rho^{2\gamma-2} r^6} \right\} dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \rho^{2\gamma-2} r^4 |D_x \rho|^2 dx \right)^{\frac{1}{2}} \\
&\quad + \sup_{x_1 \leq x \leq x_2} \left| \frac{1}{\rho^{2\gamma-2}} \right| \int_{x_1}^{x_2} \rho^{2\gamma-2} r^4 |D_x \rho|^2 dx \\
&\quad + \left(\int_{x_1}^{x_2} \frac{1}{\rho^{4\gamma-4} r^4} dx \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} \rho^{4\gamma-2} r^8 |D_x^2 \rho|^2 dx \right)^{\frac{1}{2}}
\end{aligned}$$

This concludes the proof of the lemma. \square

Thus the a priori estimates can be closed at this point. In the following sections, being inspired by the a priori estimates, we construct local in time strong solutions to the Navier-Stokes-Poisson system.

5 Approximate Scheme

Strong solutions to the free boundary problem to (1.2) with given initial data $\rho(0, r) = \rho_{in}(r)$ and $u(0, r) = u_{in}(r)$, and boundary conditions (1.5), (1.6), (1.7) are constructed by an approximate scheme, where an approximate velocity is obtained by solving a parabolic linear PDE in Lagrangian coordinates and the following approximate density profile is defined by the flow generated by the approximate velocity. Due to the coordinate singularity at the origin in Lagrangian formulation, the corresponding Eulerian formulation is invoked and both Lagrangian and Eulerian estimates are obtained.

Let initial data ρ_{in}, u_{in} be given: for $0 \leq r \leq R$, $\rho_{in}(r), u_{in}(r)$ satisfy

$$\mathcal{E}|_{(\rho_{in}, u_{in})} \leq A \text{ for some } A > 0; \rho_{in}(R) = 0; \rho_{in}(r) > 0 \text{ for } 0 \leq r < R; \int_0^R \rho_{in} r^2 dr = 1. \quad (5.1)$$

Starting with these initial data, in the following, we describe an approximate scheme that leads to approximations (ρ^n, u^n, r^n) for all $n \geq 0$. Subsequently, we study the existence, uniqueness, and regularity of them. Approximations are shown to be uniformly bounded under the energy space characterized by \mathcal{E} , and finally, a strong solution is obtained by taking the limit.

For given initial data ρ_{in} and u_{in} satisfying (5.1) in Eulerian coordinates, first introduce a Lagrangian variable x as follows:

$$x \equiv \int_0^r \rho_{in} s^2 ds, \quad 0 \leq x \leq 1.$$

Then ρ_{in} and u_{in} , denoted by ρ^0 and u^0 respectively, can be regarded as functions of x . Define r^0 by

$$r^0 \equiv \{3 \int_0^x \frac{1}{\rho^0} dy\}^{\frac{1}{3}}.$$

We would like to define the sequence $\{\rho^n, r^n, u^n\}$ inductively for all $n \geq 0$. Suppose that ρ^n, r^n , and u^n are known functions. Firstly, consider the following linear partial differential equations for u^{n+1} :

$$D_t u^{n+1} - \mu D_x (\rho^n (r^n)^4 D_x u^{n+1}) + \mu \frac{2u^{n+1}}{\rho^n (r^n)^2} = -(r^n)^2 D_x p^n - \frac{4\pi x}{(r^n)^2} \quad (5.2)$$

with the initial data $u^{n+1}(0, x) = u_{in}$ and boundary conditions

$$u^{n+1}(t, 0) = 0 \text{ and } (\mu \rho^n (r^n)^2 D_x u^{n+1} - p^n)(t, 1) = 0.$$

(5.2) is a parabolic-type equation with degenerate coefficients at $x = 0, 1$, but the singularity is either coordinate singularity or point singularity and hence the existence, uniqueness, and regularity follow from the classical theory. Note that by the change of variables

$$D_x = \frac{1}{\rho^n (r^n)^2} \partial_{r^n} \text{ and } D_t = \partial_t + (D_t r^n) \partial_{r^n}, \quad (5.3)$$

(5.2) can be written in Eulerian coordinates (t, r^n) as follows:

$$\partial_t u^{n+1} + (D_t r^n) \partial_{r^n} u^{n+1} - \mu \frac{1}{\rho^n (r^n)^2} \partial_{r^n} ((r^n)^2 \partial_{r^n} u^{n+1}) + \mu \frac{2u^{n+1}}{\rho^n (r^n)^2} = -\frac{1}{\rho^n} \partial_{r^n} p^n - \frac{4\pi x}{(r^n)^2} \quad (5.4)$$

where $D_t r^n = -\frac{1}{(r^n)^2} \int_0^x \frac{D_t \rho^n}{(\rho^n)^2} dy$ and $\partial_t x = -\rho^n (r^n)^2 D_t r^n$; $\partial_{r^n} x = \rho^n (r^n)^2$. Next, imitating the

formula (2.1), define ρ^{n+1} by

$$\rho^{n+1}(t, x) \equiv \rho^0 \exp\left\{-\int_0^t \rho^n(r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n} d\tau\right\}. \quad (5.5)$$

It is straightforward to check that $\rho^{n+1}(t, x)$ satisfies the following equation:

$$D_t \rho^{n+1} + \{\rho^n(r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}\} \rho^{n+1} = 0. \quad (5.6)$$

It reads in Eulerian coordinates as follows:

$$\partial_t \rho^{n+1} + (D_t r^n) \partial_{r^n} \rho^{n+1} + \rho^{n+1} \{\partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n}\} = 0 \quad (5.7)$$

Lastly, we define r^{n+1} by

$$r^{n+1} \equiv \{3 \int_0^x \frac{1}{\rho^{n+1}} dy\}^{\frac{1}{3}}. \quad (5.8)$$

We need to make sure (5.2) is solvable and (5.5) and (5.8) make sense in an appropriate sense. First, we study (5.2) in a weak formulation in Lagrangian coordinates, and establish the regularity of weak solution. Interior regularity is standard because (5.2) is parabolic bounded away from the boundary, while boundary regularity is obtained with weights in the form of integrals. Once one has regularity of u^{n+1} with respect to ρ^n, r^n coefficients, one can check ρ^{n+1}, r^{n+1} are well-defined. Eulerian regularity easily follows since (5.2) and (5.4) are equivalent in the interiors. We remark that for Eulerian regularity, $D_t r^n \partial_{r^n} \rho^{n+1}$ related terms in (5.4) can be taken care of by integrating by parts, since $D_t r^n$ is more regular. The most important task is to derive the uniform bound on n so that we may conclude that the limit functions are a desired solution.

The rest of this section is devoted to studying weak solutions of above approximate equations in the line of existence, uniqueness, and regularity. We use Galerkin's method well-illustrated in [4]. The difference is that the space we will work is not a typical Sobolev space, but inherited from the special structure of target equations. The first purpose is to show the existence of weak solution u^{n+1} to (5.2) for given ρ^n, r^n and u^n . Without confusion, we will drop the index n from now on. We assume that we have as much regularity of ρ and r as needed. In particular, we keep in mind the behavior of stationary solutions:

$$\rho_0 \sim (1-x)^{\frac{1}{7}} \text{ if } x \sim 1 \text{ and } r \sim x^{\frac{1}{3}} \text{ if } r \sim 0.$$

We start with Lagrangian equation (5.2):

$$D_t u - \mu D_x (\rho r^4 D_x u) + \mu \frac{2u}{\rho r^2} = -r^2 D_x p - \frac{4\pi x}{r^2}$$

Firstly, we define the notion of weak solution. To do so, a Hilbert space H is introduced:

$$H = Cl\{u \in C^\infty(0, 1) : \int_0^1 \rho r^4 |D_x u|^2 + \frac{2u^2}{\rho r^2} dx < \infty, u(0) = 0\}$$

It is straightforward to check $H \subset L^2(0, 1)$.

Definition 5.1. We say $u \in L^2(0, T; H)$ with $u' \in L^2(0, T; H^*)$ is a weak solution of (5.2) provided

$$\int_0^1 u' v dx + \mu \int_0^1 \rho r^4 D_x u D_x v dx + \mu \int_0^1 \frac{2uv}{\rho r^2} dx = \int_0^1 r^2 p D_x v dx + \int_0^1 \left(\frac{2p}{\rho r} - \frac{4\pi x}{r^2}\right) v dx$$

for each $v \in H$ and a.e. time $0 \leq t \leq T$, and $u(0) = u_{in}$. H^* is the dual space of H and $' = D_t$.

Lemma 5.2. Assume $u_{in} \in L^2$, $\rho^{-\frac{1}{2}}p \in L^2(0, T; L^2)$, and $\rho^{\frac{1}{2}}r^{-1}x \in L^2(0, T; L^2)$. There exist a unique weak solution $u \in L^2(0, T; H)$ with $u' \in L^2(0, T; H^*)$ to (5.2). Furthermore, there exists a constant $C_\mu > 0$ such that

$$\max_{0 \leq t \leq T} \|u(t)\|_{L^2} + \|u\|_{L^2(0, T; H)} + \|u'\|_{L^2(0, T; H^*)} \leq C_\mu \left\{ \|u_{in}\|_{L^2} + \left\| \frac{p}{\sqrt{\rho}} \right\|_{L^2(0, T; L^2)} + \left\| \sqrt{\rho} \frac{4\pi x}{r} \right\|_{L^2(0, T; L^2)} \right\}. \quad (5.9)$$

Proof. Let $w_k = w_k(x)$ ($k = 1, 2, \dots$) be an orthogonal basis of H and orthonormal in L^2 when $t = 0$, i.e., $\rho(0) = \rho_{in}$ and $r(0) = r_{in}$. Then $\{w_k\}$ forms a basis of H for $0 \leq t \leq T$, where T is sufficiently small, due to smoothness of ρ, r . Fix a positive integer m . We seek a function $u_m : [0, T] \rightarrow H$ of the form

$$u_m(t) = \sum_{k=1}^m d_m^k(t) w_k, \quad (5.10)$$

where

$$d_m^k(0) = \int_0^1 u_{in} w_k dx \quad (k = 1, \dots, m) \quad (5.11)$$

and for each $k = 1, \dots, m$, $0 \leq t \leq T$,

$$\int_0^1 u_m' w_k dx + \mu \int_0^1 \rho r^4 D_x u_m D_x w_k dx + \mu \int_0^1 \frac{2u_m w_k}{\rho r^2} dx = \int_0^1 r^2 p D_x w_k dx + \int_0^1 \left(\frac{2p}{\rho r} - \frac{4\pi x}{r^2} \right) w_k dx. \quad (5.12)$$

Claim 1. For each m , there exists a unique function u_m of the form (5.10) satisfying (5.11) and (5.12).

Proof of Claim 1: Note that

$$\int_0^1 u_m'(t) w_k dx = d_m^{k'}(t), \quad \mu \int_0^1 \rho r^4 D_x u_m D_x w_k dx + \mu \int_0^1 \frac{2u_m w_k}{\rho r^2} dx = \sum_{l=1}^m e^{kl}(t) d_m^l(t)$$

where $e^{kl}(t) = \mu \int_0^1 \rho r^4 D_x w_l D_x w_k dx + \mu \int_0^1 \frac{2w_l w_k}{\rho r^2} dx$. Let us write

$$f^k(t) = \int_0^1 r^2 p D_x w_k dx + \int_0^1 \left(\frac{2p}{\rho r} - \frac{4\pi x}{r^2} \right) w_k dx.$$

Then (5.12) becomes the linear system of ODEs

$$d_m^{k'}(t) + \sum_{l=1}^m e^{kl}(t) d_m^l(t) = f^k(t), \quad (5.13)$$

subject to the initial condition (5.11). According to the standard existence theory for ordinary differential equations, there exists a unique absolutely continuous functions $d_m^k(t)$ satisfying (5.11) and (5.13) for a.e. $0 \leq t \leq T$. And then u_m defined by (5.10) solves (5.12).

Claim 2. The following energy estimates hold:

$$\begin{aligned} \max_{0 \leq t \leq T} \|u_m(t)\|_{L^2} + \|u_m\|_{L^2(0, T; H)} + \|u_m'\|_{L^2(0, T; H^*)} \\ \leq C_\mu \left\{ \|u_{in}\|_{L^2} + \left\| \frac{p}{\sqrt{\rho}} \right\|_{L^2(0, T; L^2)} + \left\| \sqrt{\rho} \frac{4\pi x}{r} \right\|_{L^2(0, T; L^2)} \right\}, \end{aligned} \quad (5.14)$$

where C_μ is a constant independent of m .

Proof of Claim 2: Multiplying (5.12) by d_m^k and summing over k , we get

$$\begin{aligned} \int_0^1 u'_m u_m dx + \mu \int_0^1 \rho r^4 |D_x u_m|^2 + \frac{2u_m^2}{\rho r^2} dx &= \int_0^1 r^2 p D_x u_m dx + \int_0^1 \left(\frac{2p}{\rho r} - \frac{4\pi x}{r^2} \right) u_m dx \\ \Rightarrow \frac{1}{2} \frac{d}{dt} \|u_m\|_{L^2}^2 + \mu \int_0^1 \rho r^4 |D_x u_m|^2 + \frac{2u_m^2}{\rho r^2} dx &\leq \frac{\mu}{2} \int_0^1 \rho r^4 |D_x u_m|^2 + \frac{2u_m^2}{\rho r^2} dx \\ &\quad + \frac{2}{\mu} \int_0^1 A^2 \rho^{2\gamma-1} + \rho \frac{16\pi^2 x^2}{r^2} dx \end{aligned}$$

By taking t -integral,

$$\begin{aligned} \|u_m\|_{L^2}^2 + \mu \int_0^t \int_0^1 \rho r^4 |D_x u_m|^2 + \frac{2u_m^2}{\rho r^2} dx ds &\leq \|u_m(0)\|_{L^2}^2 + \frac{4}{\mu} \int_0^t \int_0^1 A^2 \rho^{2\gamma-1} + \rho \frac{16\pi^2 x^2}{r^2} dx ds \\ \Rightarrow \max_{0 \leq t \leq T} \|u_m(t)\|_{L^2}^2 + \mu \int_0^T \int_0^1 \rho r^4 |D_x u_m|^2 + \frac{2u_m^2}{\rho r^2} dx ds &\leq \|u_{in}\|_{L^2}^2 + \frac{4}{\mu} \left(\left\| \frac{p}{\sqrt{\rho}} \right\|_{L^2(0,T;L^2)}^2 + \left\| \sqrt{\rho} \frac{4\pi x}{r} \right\|_{L^2(0,T;L^2)}^2 \right) \end{aligned}$$

Fix $v \in H$ with $\|v\|_H \leq 1$. Write $v = v^1 + v^2$, $v^1 \in \text{span}\{w_k\}_{k=1}^m$, and $\int_0^1 v^2 w_k dx = 0$, $k = 1, \dots, m$. Since $\{w_k\}$ is orthogonal, $\|v^1\|_H \leq \|v\|_H \leq 1$. From (5.12),

$$\begin{aligned} \int_0^1 u'_m v dx &= \int_0^1 u'_m v^1 dx = \int_0^1 r^2 p D_x v^1 dx + \int_0^1 \left(\frac{2p}{\rho r} - \frac{4\pi x}{r^2} \right) v dx - \mu \int_0^1 \rho r^4 D_x u_m D_x v^1 + \frac{2u_m v^1}{\rho r^2} dx \\ &\leq C \left\{ \left\| \frac{p}{\sqrt{\rho}} \right\|_{L^2} + \left\| \sqrt{\rho} \frac{4\pi x}{r} \right\|_{L^2} + \mu \|u_m\|_H \right\} \end{aligned}$$

Hence,

$$\begin{aligned} \|u'_m\|_{H^*} &\leq C \left\{ \left\| \frac{p}{\sqrt{\rho}} \right\|_{L^2} + \left\| \sqrt{\rho} \frac{4\pi x}{r} \right\|_{L^2} + \mu \|u_m\|_H \right\} \text{ and} \\ \int_0^T \|u'_m\|_{H^*} dt &\leq C_\mu \left\{ \|u_{in}\|_{L^2} + \left\| \frac{p}{\sqrt{\rho}} \right\|_{L^2(0,T;L^2)} + \left\| \sqrt{\rho} \frac{4\pi x}{r} \right\|_{L^2(0,T;L^2)} \right\}. \end{aligned}$$

Now we pass to limits as $m \rightarrow \infty$. According to energy estimates, $\{u_m\}_{m=1}^\infty$ is bounded in $L^2(0, T; H)$, $\{u'_m\}$ bounded in $L^2(0, T; H^*)$ and therefore, there exist a subsequence $\{u_{m_l}\} \subset \{u_m\}$ and functions $u \in L^2(0, T; H)$, $u' \in L^2(0, T; H^*)$ such that

$$u_{m_l} \rightharpoonup u \text{ weakly in } L^2(0, T; H), \quad u'_{m_l} \rightharpoonup u' \text{ weakly in } L^2(0, T; H^*).$$

Fix N , and consider $v \in C^1(0, T; H)$ having the form of $v(t) = \sum_{k=1}^N d^k(t) w_k$. Choose $m \geq N$,

$$\int_0^T \int_0^1 u'_m v dx dt + \mu \int_0^T \int_0^1 \rho r^4 D_x u_m D_x v + \frac{2u_m v}{\rho r^2} dx dt = \int_0^T \int_0^1 r^2 p D_x v dx + \int_0^T \int_0^1 \left(\frac{2p}{\rho r} - \frac{4\pi x}{r^2} \right) v dx dt.$$

Setting $m = m_l$, pass to weak limits to get for all $v \in L^2(0, T; H)$,

$$\int_0^T \int_0^1 u' v dx dt + \mu \int_0^T \int_0^1 \rho r^4 D_x u D_x v + \frac{2uv}{\rho r^2} dx dt = \int_0^T \int_0^1 r^2 p D_x v dx + \int_0^T \int_0^1 \left(\frac{2p}{\rho r} - \frac{4\pi x}{r^2} \right) v dx dt.$$

In particular,

$$\int_0^1 u' v dx + \mu \int_0^1 \rho r^4 D_x u D_x v + \frac{2uv}{\rho r^2} dx = \int_0^1 r^2 p D_x v dx + \int_0^1 \left(\frac{2p}{\rho r} - \frac{4\pi x}{r^2} \right) v dx, \quad \forall v \in H \text{ a.e. } t.$$

This assures the existence of weak solution to (5.2). Uniqueness of weak solutions easily follows from

energy estimates: let u_1, u_2 be two weak solutions with the same initial data, then $u \equiv u_1 - u_2$ satisfies the following:

$$\int_0^T \int_0^1 u' v dx dt + \mu \int_0^T \int_0^1 \rho r^4 D_x u D_x v + \frac{2uv}{\rho r^2} dx dt = 0, \quad \forall v \in L^2(0, T; H).$$

Choose $v = u \in L^2(0, T; H)$, and we get

$$\frac{1}{2} \|u\|_{L^2}^2 + \mu \int_0^T \int_0^1 \rho r^4 |D_x u|^2 + \frac{2u^2}{\rho r^2} dx dt = \frac{1}{2} \|u(0)\|_{L^2}^2 = 0,$$

and hence $u = 0$ a.e. \square

Next, we try to get the regularity of the weak solution u obtained in the above. First, we establish Lagrangian regularity. Eulerian regularity is obtained by the change of variable (5.3) in the integral form. One can study the weak solution by the same Galerkin method in Eulerian formulation, but we skip the details here. The next lemma regards time regularity.

Lemma 5.3. Assume $\sup_{0 < x < 1} |\frac{\rho'}{\rho}| \leq C_1$ and $\sup_{0 < x < 1} |\frac{r'}{r}| \leq C_2$ for $0 \leq t \leq T$. In addition, assume $u_{in} \in H$ and $r^2 D_x p + \frac{4\pi x}{r^2} \in L^2(0, T; L^2)$. Then $u \in L^\infty(0, T; H)$, $u' \in L^2(0, T; L^2)$ with the estimate

$$\begin{aligned} \int_0^T \|u'\|_{L^2}^2 dt + \sup_{0 \leq t \leq T} \mu \|u(t)\|_H^2 &\leq C \{ \|u_{in}\|_H^2 + \|u_{in}\|_{L^2}^2 \\ &+ \|r^2 D_x p + \frac{4\pi x}{r^2}\|_{L^2(0, T; L^2)}^2 + \|\frac{p}{\sqrt{\rho}}\|_{L^2(0, T; L^2)}^2 + \|\sqrt{\rho} \frac{4\pi x}{r}\|_{L^2(0, T; L^2)}^2 \}. \end{aligned} \quad (5.15)$$

Proof. Again, Galerkin method is used. We start with (5.2). Let $u_m(t) = \sum_{k=1}^m d_m^k(t) w_k$. Multiplying (5.12) by d_m^k and summing over k to get

$$\begin{aligned} \int_0^1 u'_m u'_m dx + \mu \int_0^1 \rho r^4 D_x u_m D_x u'_m + \frac{2u_m u'_m}{\rho r^2} dx &= \int_0^1 r^2 p D_x u'_m dx + \int_0^1 (\frac{2p}{\rho r} - \frac{4\pi x}{r^2}) u'_m dx \\ &= - \int_0^1 (r^2 D_x p + \frac{4\pi x}{r^2}) u'_m dx \\ \Rightarrow \|u'_m\|_{L^2}^2 + \frac{d}{dt} \frac{\mu}{2} \int_0^1 \rho r^4 |D_x u_m|^2 + \frac{2u_m^2}{\rho r^2} dx &\leq \frac{1}{2} \|u'_m\|_{L^2}^2 + \frac{1}{2} \|r^2 D_x p + \frac{4\pi x}{r^2}\|_{L^2}^2 \\ &\quad + \underbrace{\frac{\mu}{2} \int_0^1 (\rho' r^4 + 4\rho r^3 r') (D_x u_m)^2 - (\frac{2\rho'}{\rho^2 r^2} + \frac{4r'}{\rho r^3}) u_m^2 dx}_{**} \end{aligned}$$

Since $|\frac{\rho'}{\rho}| \leq C_1$ and $|\frac{r'}{r}| \leq C_2$, $** \leq 2\mu(C_1 + C_2) \|u_m\|_H^2$, we get

$$\begin{aligned} \int_0^T \|u'_m\|_{L^2}^2 dt + \sup_{0 \leq t \leq T} \mu \int_0^1 \rho r^4 |D_x u_m|^2 + \frac{2u_m^2}{\rho r^2} dx \\ \leq \mu \int_0^1 \rho_{in} r_{in}^4 |D_x u_{in}|^2 + \frac{2u_{in}^2}{\rho_{in}^2} dx + \int_0^T \|r^2 D_x p + \frac{4\pi x}{r^2}\|_{L^2}^2 dt + 4\mu(C_1 + C_2) \int_0^T \|u_m\|_H^2 dt \\ \leq C \{ \|u_{in}\|_H^2 + \|u_{in}\|_{L^2}^2 + \|r^2 D_x p + \frac{4\pi x}{r^2}\|_{L^2(0, T; L^2)}^2 + \|\frac{p}{\sqrt{\rho}}\|_{L^2(0, T; L^2)}^2 + \|\sqrt{\rho} \frac{4\pi x}{r}\|_{L^2(0, T; L^2)}^2 \} \end{aligned}$$

Pass to limit $m \rightarrow \infty$. (5.15) holds and the lemma follows. \square

Now we would like to establish regularity in x variable. Note that (5.2) is one-dimensional parabolic equation as long as x is bounded away from the boundary and hence interior regularity can be easily shown by using standard different quotients method (see Section 7.1 in [4]), i.e. $u \in H_{\text{loc}}^2(0, 1)$. Here H^2 represents the usual Sobolev space. Recall that

$$\mu \int_0^1 \rho r^4 D_x u D_x v dx + \mu \int_0^1 \frac{2uv}{\rho r^2} dx = \int_0^1 r^2 p D_x v dx + \int_0^1 \left(\frac{2p}{\rho r} - \frac{4\pi x}{r^2} - u' \right) v dx, \quad \forall v \in H. \quad (5.16)$$

We can now integrate by parts in (5.16) by approximating H with $v \in C_c^\infty(0, 1) \subset H$:

$$-\mu \int_0^1 D_x(\rho r^4 D_x u) v dx + \mu \int_0^1 \frac{2uv}{\rho r^2} dx = - \int_0^1 \left(r^2 D_x p + \frac{4\pi x}{r^2} + u' \right) v dx, \quad \forall v \in C_c^\infty(0, 1),$$

Therefore, u actually solves the PDE for a.e., and the following estimate can be obtained from the equation:

$$\begin{aligned} \frac{\mu^2}{2} \int_0^1 \rho r^2 |r^2 D_x(\rho r^2 D_x u)|^2 dx &\leq \int_0^1 \rho r^2 |u'|^2 dx + \mu^2 \int_0^1 \frac{4u^2}{\rho r^2} + 4\rho r^4 |D_x u|^2 dx \\ &\quad + \int_0^1 \rho r^2 |r^2 D_x p + \frac{4\pi x}{r^2}|^2 dx \end{aligned} \quad (5.17)$$

Note that L^1 estimate for $D_x(\rho r^2 D_x u)$ can be obtained from the equation. Next we claim that u satisfies the boundary condition $(\rho r^2 D_x u)(t, 1) = 0$ for $0 \leq t \leq T$ in the trace sense. Note that $D_x(\rho r^2 D_x u)$ is in $L^1(r^2 dx)$ and therefore by the Trace theorem, $\rho r^2 D_x u$ at $x = 1$ is well-defined. Thus, in (5.16), we can integrate by parts up to boundary for the first term to get: recall $v(0) = 0$,

$$\mu(\rho r^4 D_x u v)(t, 1) - \mu \int_0^1 D_x(\rho r^4 D_x u) v dx + \mu \int_0^1 \frac{2uv}{\rho r^2} dx = - \int_0^1 \left(r^2 D_x p + \frac{4\pi x}{r^2} + u' \right) v dx, \quad \forall v \in H$$

We already know that u solves the PDE a.e. and so, all other terms vanish, and therefore we obtain $(\rho r^2 D_x u)(t, 1) = 0$, the desired boundary condition. We have proven spatial regularity of u^{n+1} :

Lemma 5.4. *The weak solution u solves (5.2) and it is regular: $u \in H_{\text{loc}}^2(0, 1)$ and weak boundary regularity is given by the estimate (5.17). Moreover, u satisfies the boundary condition.*

Higher regularity: Now let us build the higher regularity based on the previous regularity assertion. Set $\tilde{u} = u'$. Differentiating (5.2) with respect to t , one can check that \tilde{u} is the unique weak solution of

$$D_t \tilde{u} - \mu D_x(\rho r^4 D_x \tilde{u}) + \mu \frac{2\tilde{u}}{\rho r^2} = -D_t(r^2 D_x p + \frac{4\pi x}{r^2}) + \mu D_x(D_t(\rho r^4) D_x u) + \mu D_t(\frac{2}{\rho r^2}) u$$

with given compatible, well-prepared initial data. Note that each term in the RHS is either known function or lower order derivative terms. In the weak formulation, the spatial derivative of p and $D_t(\rho r^4) D_x u$ should be moved to test functions.

Lemma 5.5. *The weak solution u attains higher regularity as long as initial data u_{in} as well as coefficients ρ^n, r^n are regular: $u \in H_{\text{loc}}^4(0, 1)$ and weak boundary regularity is available in the integral form.*

Proof. By the same argument as before, we get the regularity of \tilde{u} with the following energy estimates:

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\tilde{u}\|_{L^2} + \|\tilde{u}\|_{L^2(0, T; H)} + \|\tilde{u}'\|_{L^2(0, T; H^*)} &\leq C(\|\tilde{f}_1\|_{L^2} + \|\tilde{u}_{in}\|_{L^2}), \\ \|\tilde{u}'\|_{L^2(0, T; L^2)} + \sup_{0 \leq t \leq T} \sqrt{\mu} \|\tilde{u}\|_H &\leq C(\|\tilde{u}_{in}\|_H + \|\tilde{u}_{in}\|_{L^2} + \|\tilde{f}_2\|_{L^2(0, T; L^2)}), \end{aligned}$$

where \tilde{f}_1 is a function of $\rho, r, u, D_x u D_t \rho, D_t r$ and \tilde{f}_2 is a function of \tilde{f}_1 and $D_x \rho$. It is routine to check $\tilde{u} \in H_{\text{loc}}^2(0, 1)$ and hence, \tilde{u} solves the above PDE for a.e., and the following estimate (weak boundary regularity) is obtained:

$$\frac{\mu^2}{2} \int_0^1 \rho^2 (r^2 D_x (\rho^2 D_x \tilde{u}))^2 dx \leq \int_0^1 \rho^2 (\tilde{u}')^2 dx + \mu^2 \int_0^1 \frac{4\tilde{u}^2}{\rho r^2} + 4\rho r^4 |D_x \tilde{u}|^2 dx + \int_0^1 \rho^2 (\tilde{f}_2)^2 dx$$

In the same line, letting $\tilde{u} = \tilde{u}'$, we get the desired regularity together with the estimates:

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\tilde{u}\|_{L^2} + \|\tilde{u}\|_{L^2(0, T; H)} + \|\tilde{u}'\|_{L^2(0, T; H^*)} &\leq C(\|\tilde{f}_1\|_{L^2} + \|\tilde{u}_{in}\|_{L^2}) \\ \|\tilde{u}'\|_{L^2(0, T; L^2)} + \sup_{0 \leq t \leq T} \sqrt{\mu} \|\tilde{u}\|_H &\leq C(\|\tilde{u}_{in}\|_H + \|\tilde{u}_{in}\|_{L^2} + \|\tilde{f}_2\|_{L^2(0, T; L^2)}) \\ \mu^2 \int_0^1 \rho^2 (r^2 D_x (\rho^2 D_x \tilde{u}))^2 dx &\leq \int_0^1 \rho^2 (\tilde{u}')^2 dx + \mu^2 \int_0^1 \frac{4\tilde{u}^2}{\rho r^2} + 4\rho r^4 |D_x \tilde{u}|^2 dx + \int_0^1 \rho^2 (\tilde{f}_2)^2 dx \end{aligned}$$

Now it remains to investigate the spatial regularity regarding $D_x^3 u$ and $D_x^4 u$. Again, it is straightforward to see $u \in H_{\text{loc}}^4$. To derive boundary estimates, let us go back to (5.2). Since ρ vanishes at the boundary with certain rate, we need some weight depending on ρ in order to control the spatial derivatives of ρ and lower order derivative terms. Set the RHS of (5.2). Firstly, multiply by ρr , differentiate with respect to x , we get

$$\mu D_x (\rho r^3 D_x (\rho^2 D_x u)) = D_x (\rho r D_t u) - D_x (\rho r f) - 2\mu D_x (\rho r^3 D_x (\frac{u}{r})) \quad (5.18)$$

$$* = D_x (\rho r^2 D_x u) - D_x (\frac{u}{r}) = D_x (\rho r^2 D_x u) - \frac{D_x u}{r} + \frac{u}{\rho r^4}$$

In the view of the previous weak boundary regularity, we obtain the following estimate:

$$\begin{aligned} \frac{\mu^2}{2} \int_0^1 \rho^6 |D_x (\rho r^3 D_x (\rho^2 D_x u))|^2 dx &\leq \int_0^1 \rho^6 (D_x (\rho r D_t u))^2 dx + \int_0^1 \rho^6 (D_x (\rho r f))^2 dx \\ &\quad + 4\mu^2 \int_0^1 \rho^6 (D_x (\rho r^3 D_x (\frac{u}{r})))^2 dx \end{aligned}$$

Note that two other terms in the RHS are also bounded. Multiply (5.18) by ρr^3 , differentiate in x , and square each term to get

$$\frac{\mu^2}{2} \int_0^1 \rho^6 |D_x (\rho r^3 D_x (\rho^3 D_x (\rho^2 D_x u)))|^2 dx \leq C$$

This completes the Lagrangian regularity of u . □

We remark that the boundary regularity is weak in a sense that r, ρ as weight functions vanish at $x = 0, 1$ respectively. Due to the interior Lagrangian regularity of u and equivalence of (5.2) and (5.4) away from the boundary, u also solves the PDE (5.4) for a.e. The vacuum boundary in Eulerian coordinates $R^n(t)$ is defined by $(3 \int_0^1 \frac{1}{\rho^n} dy)^{\frac{1}{3}}$ and $x = 0$ corresponds to $r^n = 0$. Corresponding Eulerian regularity can be obtained by the change of variable (5.3). We omit the details here. In the next section, we provide the detailed energy estimates in both Lagrangian and Eulerian coordinates with cutoff functions.

Thus u^{n+1} and therefore ρ^{n+1}, r^{n+1} are all well-defined. They are regular in the classical sense away from the boundary. Motivated by the a priori estimates, in the following section, we study the behavior u^n, ρ^n, r^n under the energy \mathcal{E} . It provides not only the regularity of ρ, u up to the boundary but also the uniform bounds on n .

6 Local in time Strong Solutions

Now we would like to obtain a uniform estimate of u^n, ρ^n, r^n on n , which assures the existence of limit functions u, ρ, r . From (5.5), in order to get a uniform bound of approximate densities, one needs to control L^∞ -norm of $\rho^n(r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}$. For given T sufficiently small, define M^{n+1} by

$$M^{n+1} \equiv \sup_{0 \leq t \leq T; 0 \leq x \leq 1} |\rho^n(r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}|.$$

Our first goal is to get a uniform bound of M^{n+1} in the view of

$$|\rho^n(r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}| \leq \int_0^1 |\rho^n(r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}| dx + \int_0^1 |D_x(\rho^n(r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n})| dx.$$

Note that the second term of the RHS may not be defined due to the singularity of the origin. It is desirable to introduce cutoff functions and to make use of both Lagrangian and Eulerian estimates. To do so, we first look at the energy estimates of u^{n+1} (t -derivatives first). We use χ and ζ as cutoff functions as in the a priori estimates. Before going any further, define the separated $(n+1)$ -th energies and the $(n+1)$ -th dissipation, resembling \mathcal{E} and \mathcal{D} in Section 1, as follows:

$$\begin{aligned} \mathcal{F}^{n+1}(t) &\equiv \mathcal{F}_L^{n+1}(t) + \mathcal{F}_E^{n+1}(t) \\ &\equiv \frac{1}{2} \sum_{i=0}^3 \int_{x_0}^1 \chi |D_t^i u^{n+1}|^2 dx + \frac{\mu}{2} \sum_{i=0}^2 \int_{x_0}^1 \chi \{ \rho^n(r^n)^4 |D_x D_t^i u^{n+1}|^2 + \frac{2|D_t^i u^{n+1}|^2}{\rho^n(r^n)^2} \} dx \\ &\quad + \frac{1}{2} \sum_{i=0}^3 \int_0^{r_2-d} \zeta \rho^n |\partial_t^i u^{n+1}|^2 (r^n)^2 dr^n + \frac{\mu}{2} \sum_{i=0}^2 \int_0^{r_2-d} \zeta \{ |\partial_{r^n} \partial_t^i u^{n+1}|^2 + \frac{2|\partial_t^i u^{n+1}|^2}{(r^n)^2} \} (r^n)^2 dr^n \end{aligned}$$

$$\begin{aligned} \mathcal{H}^{n+1}(t) &\equiv \mathcal{H}_L^{n+1}(t) + \mathcal{H}_E^{n+1}(t) \\ &\equiv \frac{1}{\gamma-1} \int_{x_0}^1 \chi (\rho^{n+1})^{\gamma-1} dx + \frac{1}{2} \sum_{i=0}^2 \int_{x_0}^1 \chi (\rho^{n+1})^{2\gamma-2} r^4 |D_x D_t^i \rho^{n+1}|^2 dx \\ &\quad + \frac{1}{2} \sum_{i=0}^1 \int_{x_0}^1 \chi (\rho^{n+1})^{4\gamma-2} r^8 |D_x^2 D_t^i \rho^{n+1}|^2 dx + \frac{1}{2} \int_{x_0}^1 \chi (\rho^{n+1})^{8\gamma-2} r^{12} |D_x^3 \rho^{n+1}|^2 dx \\ &\quad + \frac{1}{2} \sum_{0 \leq i+j \leq 3} \int_0^{r_2-d} \zeta (\rho^{n+1})^{\gamma-2} |\partial_{r^n}^i \partial_t^j \rho^{n+1}| (r^n)^2 dr^n \end{aligned}$$

$$\begin{aligned} \mathcal{D}^{n+1}(t) &\equiv \mu \sum_{i=0}^3 \int_{x_0}^1 \chi \{ \rho^n(r^n)^4 |D_x D_t^i u^{n+1}|^2 + \frac{2|D_t^i u^{n+1}|^2}{\rho^n(r^n)^2} \} dx + \sum_{i=1}^3 \int_{x_0}^1 \chi |D_t^i u^{n+1}|^2 dx \\ &\quad + \mu \sum_{i=0}^3 \int_0^{r_2-d} \zeta \{ |\partial_{r^n} \partial_t^i u^{n+1}|^2 + \frac{2|\partial_t^i u^{n+1}|^2}{(r^n)^2} \} (r^n)^2 dr^n + \sum_{i=1}^3 \int_0^{r_2-d} \zeta \rho^n |\partial_t^i u^{n+1}|^2 (r^n)^2 dr^n \end{aligned}$$

In the same spirit as in the a priori estimates, it is easy to show that M^n 's are bounded by \mathcal{F}^n and \mathcal{H}^n ; our main interest is to obtain a uniform bound. Note that ρ^n and r^n can be controlled by M^n

for $0 \leq t \leq T$:

$$\begin{aligned}
& \bullet \rho_{in} e^{-M^n T} \leq \rho^n \leq \rho_{in} e^{M^n T} \\
& \bullet r_{in} e^{-\frac{M^n T}{3}} \leq r^n \leq r_{in} e^{\frac{M^n T}{3}} \\
& \bullet \sup \left| \frac{D_t \rho^n}{\rho^n} \right| = |\rho^{n-1} (r^n)^2 D_x u^n + \frac{2u^n}{r^{n-1}}| \leq M^n \\
& \bullet D_t r^n = -\frac{1}{(r^n)^2} \int_0^x \frac{D_t \rho^n}{\rho^n} \frac{1}{\rho^n} dy \Rightarrow \sup \left| \frac{D_t r^n}{r^n} \right| \leq \frac{M^n}{3}
\end{aligned} \tag{6.1}$$

In particular, the particle paths $r_i^n(t)$ emanating from r_i for $i = 0, 1, 2$ which correspond to x_i can be bounded as follows:

$$r_i^n(t) \equiv \left(3 \int_0^{x_i} \frac{1}{\rho^n} dy \right)^{\frac{1}{3}} \Rightarrow r_i e^{-\frac{M^n T}{3}} \leq r_i^n(t) \leq r_i e^{\frac{M^n T}{3}}$$

We observe that for each given n and d , there exists a sufficiently small $T > 0$ such that $|e^{\frac{M^n T}{3}} - 1| < d$, and thus cutoff functions defined at the beginning of this article play the same role as in the a priori estimates. We will show that M^n 's are uniformly bounded and T does not shrink to zero. Before we state the result, let us speculate about the energy defined in the above: note that $\mathcal{F}^{n+1}(t)$ does not include some mixed, spatial derivatives of u^{n+1} and \mathcal{H}_L^{n+1} does not include pure t -derivative terms, which we need to close the estimates in the end. We show missing derivative terms can be estimated in terms of $\mathcal{F}^{n+1}(t)$ and $\mathcal{H}^n(t)$ via the equations (5.2) and (5.4). Consider the following equations:

$$\begin{aligned}
\mu D_x (\rho^n (r^n)^2 D_x u^{n+1}) &= -\mu \frac{2D_x u^{n+1}}{r^n} + \mu \frac{2u^{n+1}}{\rho^n (r^n)^4} + \frac{D_t u^{n+1}}{(r^n)^2} + D_x p^n + \frac{4\pi x}{(r^n)^4} \\
\Rightarrow \frac{\mu^2}{2} \int_{x_0}^1 \chi \rho^n |(r^n)^2 D_x (\rho^n (r^n)^2 D_x u^{n+1})|^2 dx &\leq C \frac{e^{\frac{2M^n T}{3}}}{r_0^2} \mathcal{F}_L^{n+1}(t) + C \left(\sup_{x_0 \leq x \leq 1} \rho_{in} \right) e^{M^n T} \mathcal{H}^n(t) \\
&\leq C_{in,n} (\mathcal{F}^{n+1}(t) + \mathcal{H}^n(t)),
\end{aligned}$$

where $C_{in,n}$ depends on initial data, $\mathcal{F}^n, \mathcal{H}^n$. Similarly, take D_t or D_x derivatives of the above equation to get the estimates:

$$\begin{aligned}
\frac{\mu}{2} \int_{x_0}^1 \chi \rho^n |(r^n)^2 D_x (\rho^n (r^n)^2 D_t D_x u^{n+1})|^2 dx &\leq C_{in,n} (\mathcal{F}^{n+1}(t) + \mathcal{H}^n(t)) \text{ and} \\
\mu^2 \int_{x_0}^1 \chi |\rho^n (r^n)^2 D_x (\rho^n (r^n)^3 D_x (\rho^n (r^n)^2 D_x u^{n+1}))|^2 dx &\leq C_{in,n} (\mathcal{F}^{n+1}(t) + \mathcal{H}^n(t))
\end{aligned}$$

For details, we refer to the a priori estimates. Similarly, L^2 norm of $\partial_{r^n}^2 u^{n+1}$, $\partial_t \partial_{r^n}^2 u^{n+1}$, $\partial_{r^n}^3 u^{n+1}$ can be estimated in Eulerian coordinates by using equations. For pure Lagrangian t -derivative terms, we directly use the approximate continuity equation (5.6): for $i = 1, 2, 3$,

$$\int_{x_0}^1 \chi \frac{|D_t^i \rho^{n+1}|^2}{(\rho^{n+1})^3} dx \leq C e^{(M^{n+1} + M^n)T} \mathcal{F}^{n+1}(t)$$

where we have used $|\frac{\rho^{n+1}}{\rho^n}| \leq e^{(M^{n+1} + M^n)T}$ for $0 \leq t \leq T$.

Proposition 6.1. *There exist $A_0, A_1, T^* > 0$ such that if $\mathcal{F}(0) \leq A_0, \mathcal{H}(0) \leq A_1$ and $\sup_{0 \leq t \leq T^*} \mathcal{F}^n(t) \leq 2A_0, \sup_{0 \leq t \leq T^*} \mathcal{H}^n(t) \leq 2A_1$ then*

$$\sup_{0 \leq t \leq T^*} \mathcal{F}^{n+1}(t) \leq 2A_0 \text{ and } \sup_{0 \leq t \leq T^*} \mathcal{H}^{n+1}(t) \leq 2A_1.$$

In particular, we can choose T^ small enough so that there exists $M > 0$ such that $M^n \leq M$ for all*

n , and for $0 \leq t \leq T^*$.

To prove Proposition 6.1, we establish the following chain-type energy inequalities. We start with u -part energy \mathcal{F}^{n+1} .

Lemma 6.2. *Let $\rho^{n+1}, u^{n+1}, \rho^n, r^n, u^n$ be regular approximate solutions obtained in Section 5. Then there exist constants $C_1, \dots, C_5 > 0$ such that for some positive numbers $\epsilon_1, \epsilon_2 > 0$,*

$$\frac{d}{dt} \mathcal{F}^{n+1}(t) + \frac{1}{8} \mathcal{D}^{n+1}(t) \leq C_1 \{(M^n + M^{n+1}) \mathcal{F}^{n+1}(t) + (\mathcal{F}^{n+1}(t))^2\} + \mathcal{G}^{n+1}(t) + \mathcal{OL}_1^{n+1}(t), \quad (6.2)$$

where

$$\mathcal{G}^{n+1}(t) \leq C_2 \{(1 + M^{n+1} + |M^n|^2 + \mathcal{F}^n(t)) \mathcal{H}^n(t) + (\mathcal{H}^n(t))^2\} + C_3 e^{C_4(M^n + M^{n-1})T} \mathcal{F}^n(t), \quad (6.3)$$

$$\mathcal{OL}_1^{n+1}(t) \leq C_5 \{\mathcal{F}^{n+1}(t) + \mathcal{H}^n(t) + (\mathcal{F}^{n+1}(t))^{1+\epsilon_1} + (\mathcal{H}^n(t))^{1+\epsilon_2}\}. \quad (6.4)$$

The estimate of ρ -part energy \mathcal{H}^{n+1} is given in the next lemma.

Lemma 6.3. *Let $\rho^{n+1}, u^{n+1}, \rho^n, r^n, u^n$ be regular approximate solutions obtained in Section 5. Then there exist constants $C_6, \dots, C_9 > 0$ such that for some positive numbers $\epsilon_3, \epsilon_4 > 0$,*

$$\begin{aligned} \frac{d}{dt} \mathcal{H}^{n+1}(t) \leq C_6 (M^n + M^{n+1}) \mathcal{H}^{n+1}(t) + C_7 (\mathcal{F}^{n+1}(t) + \mathcal{H}^n(t)) (C_{in} e^{C_8 M^{n+1} T} + \mathcal{H}^{n+1}(t)) \\ + \mathcal{OL}_2^{n+1}(t), \end{aligned} \quad (6.5)$$

where

$$\mathcal{OL}_2^{n+1}(t) \leq C_9 \{\mathcal{H}^{n+1}(t) + \mathcal{H}^n(t) + (\mathcal{H}^{n+1}(t))^{1+\epsilon_3} + (\mathcal{H}^n(t))^{1+\epsilon_4}\}.$$

We observe that it is enough to show the above energy inequalities to prove Proposition 6.1: Note that by (6.6) and the assumption, (6.2) can be reduced to the following:

$$\frac{d}{dt} \mathcal{F}^{n+1}(t) \leq C_{M,t} \{\mathcal{F}^{n+1}(t) + (\mathcal{F}^{n+1}(t))^2 + A_0 + A_1 + (A_0 + A_1)^2\}$$

By solving the differential inequality, one can conclude that for some A_0, A_1 and for small enough $T_1 > 0$, $\sup_{0 \leq t \leq T_1} \mathcal{F}^{n+1}(t) \leq 2A_0$. Similarly, from (6.5), one can deduce $\sup_{0 \leq t \leq T_2} \mathcal{H}^{n+1}(t) \leq 2A_1$ for small enough $T_2 > 0$. Choose $T^* \equiv \min\{T_1, T_2\}$. The moral of the proof of the above lemmas is same as in the a priori estimates, but here, we separate the energy \mathcal{E} into u -part \mathcal{F} and ρ -part \mathcal{H} ; estimate \mathcal{F} first, and in turn \mathcal{H} in accordance with the approximate scheme.

Proof. of Lemma 6.2: Now we are ready to prove (6.2). The spirit is same as in the a priori estimates. Eulerian estimates and Lagrangian estimates are to be performed concurrently. We provide rather detailed estimates to distinguish $\mathcal{F}^n, \mathcal{F}^{n+1}, \mathcal{H}^n, \mathcal{H}^{n+1}, \mathcal{G}^{n+1}, \mathcal{OL}$ etc.

Notation: In the below, we use the double underline $\underline{\underline{\quad}}$ to denote the terms that belong to \mathcal{OL}_1^{n+1} , and the under-brace $\underbrace{\quad}$ to denote the terms that we take the sup of. The rest of terms in the RHS will either contribute to $\mathcal{G}^{n+1}(t)$ or be absorbed into the dissipation in the LHS at the last step.

Here is the zeroth order estimates. Multiply (5.2) by χu^{n+1} and integrate it over x to get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi |u^{n+1}|^2 dx + \frac{3\mu}{4} \int_{x_0}^1 \chi \{ \rho^n (r^n)^4 |D_x u^{n+1}|^2 + \frac{2|u^{n+1}|^2}{\rho^n (r^n)^2} \} dx &\leq \frac{5A^2}{\mu} \int_{x_0}^1 \chi (\rho^n)^{2\gamma-1} dx \\ &+ \frac{16\pi^2}{\mu} \int_{x_0}^1 \chi \rho^n \frac{x^2}{(r^n)^2} dx + \underbrace{\int_{x_0}^{x_1} \chi' (r^n)^2 p^n u^{n+1} dx - \mu \int_{x_0}^{x_1} \chi' u^{n+1} \rho^n (r^n)^4 D_x u^{n+1} dx}_{=0} \end{aligned}$$

Multiply (5.4) by $\zeta u^{n+1} (r^n)^2$ and integrate it over r^n to get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^{r_2-d} \zeta \rho^n |u^{n+1}|^2 (r^n)^2 dr^n + \frac{3\mu}{4} \int_0^{r_2-d} \zeta \{ |\partial_{r^n} u^{n+1}|^2 + \frac{2|u^{n+1}|^2}{(r^n)^2} \} (r^n)^2 dr^n \\ \leq \frac{3A^2}{\mu} \int_0^{r_2-d} \zeta (\rho^n)^{2\gamma} (r^n)^2 dr^n + \frac{16\pi^2}{\mu} \int_0^{r_2-d} \zeta \frac{x^2}{(r^n)^2} (\rho^n)^2 (r^n)^2 dr^n \\ + \frac{2}{\mu} \int_0^{r_2-d} \zeta \underbrace{\rho^n |D_t r^n|^2}_{=0} \rho^n |u^{n+1}|^2 (r^n)^2 dr^n + \frac{1}{2} \int_0^{r_2-d} \zeta \underbrace{\frac{\partial_t \rho^n}{\rho^n}}_{=0} \rho^n |u^{n+1}|^2 (r^n)^2 dr^n \\ - \mu \underbrace{\int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) (\partial_{r^n} u^{n+1}) u^{n+1} (r^n)^2 dr^n}_{=0} + \underbrace{\int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) p^n u^{n+1} (r^n)^2 dr^n}_{=0} \end{aligned}$$

Next, we estimate one spatial derivative of u^{n+1} . Multiply (5.2) by $\chi D_t u^{n+1}$ and integrate it over x to get:

$$\begin{aligned} \frac{\mu}{2} \frac{d}{dt} \int_{x_0}^1 \chi \{ \rho^n (r^n)^4 |D_x u^{n+1}|^2 + \frac{2|u^{n+1}|^2}{\rho^n (r^n)^2} \} dx + \frac{3}{4} \int_{x_0}^1 \chi |D_t u^{n+1}|^2 dx \\ \leq \frac{\mu}{2} \int_{x_0}^1 \chi \underbrace{\frac{D_t (\rho^n (r^n)^4)}{\rho^n (r^n)^4}}_{=0} \rho^n (r^n)^4 |D_x u^{n+1}|^2 dx + \mu \int_{x_0}^1 \chi \underbrace{\rho^n (r^n)^2 D_t \left(\frac{1}{\rho^n (r^n)^2} \right)}_{=0} \frac{|u^{n+1}|^2}{\rho^n (r^n)^2} dx \\ + \int_{x_0}^1 \chi (r^n)^4 |D_x p^n|^2 dx + \frac{\mu}{4} \int_{x_0}^1 \chi \frac{|D_t u^{n+1}|^2}{\rho^n (r^n)^2} dx + \frac{1}{\mu} \int_{x_0}^1 \chi \frac{16\pi^2 x^2 \rho^n}{(r^n)^2} dx \\ - \mu \underbrace{\int_{x_0}^{x_1} \chi' \rho^n (r^n)^4 D_x u^{n+1} D_t u^{n+1} dx}_{=0} \end{aligned}$$

Multiply (5.4) by $\zeta \rho^n \partial_t u^{n+1} (r^n)^2$ and integrate it over r^n to get:

$$\begin{aligned} \frac{\mu}{2} \frac{d}{dt} \int_0^{r_2-d} \zeta \{ |\partial_{r^n} u^{n+1}|^2 + \frac{2|u^{n+1}|^2}{(r^n)^2} \} (r^n)^2 dr^n + \frac{3}{4} \int_0^{r_2-d} \zeta \rho^n |\partial_t u^{n+1}|^2 (r^n)^2 dr^n \\ \leq 6 \int_0^{r_2-d} \zeta \underbrace{\rho^n |D_t r^n|^2}_{=0} |\partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n + 6 \int_0^{r_2-d} \frac{1}{\rho^n} |\partial_{r^n} p^n|^2 (r^n)^2 dr^n \\ + 6 \int_0^{r_2-d} \zeta \frac{16\pi^2 x^2}{(r^n)^2} \rho^n dr^n - \mu \underbrace{\int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) (\partial_t u^{n+1}) (\partial_{r^n} u^{n+1}) (r^n)^2 dr^n}_{=0} \end{aligned}$$

Next, in order to estimate one temporal derivative of u^{n+1} , take D_t of (5.2), multiply it by $\chi D_t u^{n+1}$,

and integrate to get:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi |D_t u^{n+1}|^2 dx + \frac{3\mu}{4} \int_{x_0}^1 \chi \{ \rho^n(r^n)^4 |D_t D_x u^{n+1}|^2 + \frac{2|D_t u^{n+1}|^2}{\rho^n(r^n)^2} \} dx \\
& \leq 2\mu \int_{x_0}^1 \underbrace{\chi \left| \frac{D_t(\rho^n(r^n)^4)}{\rho^n(r^n)^4} \right|^2}_{\rho^n(r^n)^4} |D_x u^{n+1}|^2 dx + 8\mu \int_{x_0}^1 \underbrace{\chi |\rho^n(r^n)^2 D_t(\frac{1}{\rho^n(r^n)^2})|^2}_{\frac{1}{\rho^n(r^n)^2}} \frac{|u^{n+1}|^2}{\rho^n(r^n)^2} dx \\
& + \frac{2}{\mu} \int_{x_0}^1 \chi \frac{|D_t((r^n)^2 p^n)|^2}{\rho^n(r^n)^4} dx + \frac{2}{\mu} \int_{x_0}^1 \chi \rho^n(r^n)^2 |D_t(\frac{2p^n}{\rho^n r^n})|^2 dx + \frac{32\pi^2}{\mu} \int_{x_0}^1 \chi \rho^n(r^n)^2 |D_t(\frac{x}{(r^n)^2})|^2 dx \\
& - \mu \int_{x_0}^{x_1} \chi' \rho^n(r^n)^2 D_t u^{n+1} D_x D_t u^{n+1} dx - \mu \int_{x_0}^{x_1} \chi' D_t(\rho^n(r^n)^4) D_t u^{n+1} D_x u^{n+1} dx \\
& \quad \underline{\underline{+ \int_{x_0}^{x_1} \chi' D_t((r^n)^2 p^n) D_t u^{n+1} dx}}
\end{aligned}$$

Likewise, take ∂_t of (5.4), multiply it by $\zeta \rho^n \partial_t u^{n+1} (r^n)^2$, and integrate to get:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^{r_2-d} \zeta \rho^n |\partial_t u^{n+1}|^2 (r^n)^2 dr^n + \frac{3\mu}{4} \int_0^{r_2-d} \zeta \{ |\partial_t \partial_{r^n} u^{n+1}|^2 + \frac{2|\partial_t u^{n+1}|^2}{(r^n)^2} \} (r^n)^2 dr^n \\
& \leq \frac{2}{\mu} \int_0^{r_2-d} \zeta \underbrace{\rho^n |D_t r^n|^2}_{\rho^n} |\partial_t u^{n+1}|^2 (r^n)^2 dr^n + \frac{3}{\mu} \int_0^{r_2-d} \zeta |\partial_t p^n|^2 (r^n)^2 dr^n \\
& - \frac{1}{2} \int_0^{r_2-d} \zeta \underbrace{\frac{\partial_t \rho^n}{\rho^n}}_{\rho^n} \rho^n |\partial_t u^{n+1}|^2 (r^n)^2 dr^n - \frac{1}{\mu} \int_0^{r_2-d} \zeta \underbrace{\frac{|\partial_t(\rho^n D_t r^n)|^2}{\rho^n}}_{\rho^n} \rho^n |\partial_t u^{n+1}|^2 (r^n)^2 dr^n \\
& + \frac{\mu}{4} \int_0^{r_2-d} \zeta |\partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n + \frac{32\pi^2}{\mu} \int_0^{r_2-d} \zeta x^2 |\partial_t \rho^n|^2 dr^n + \frac{32\pi^2}{\mu} \int_0^{r_2-d} \zeta (\rho^n)^4 (r^n)^4 |D_t r^n|^2 dr^n \\
& - \mu \int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) \partial_t \partial_{r^n} u^{n+1} \partial_t u^{n+1} (r^n)^2 dr^n - \int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) \partial_t p^n \partial_t u^{n+1} (r^n)^2 dr^n \\
& \quad \underline{\underline{}}
\end{aligned}$$

Next we estimate $D_t D_x u^{n+1}$. Multiply $D_t(5.2)$ by $\chi D_t^2 u^{n+1}$ and integrate to get:

$$\begin{aligned}
& \frac{\mu}{2} \frac{d}{dt} \int_{x_0}^1 \chi \{ \rho^n(r^n)^4 |D_t D_x u^{n+1}|^2 + \frac{2|D_t u^{n+1}|^2}{\rho^n(r^n)^2} \} dx + \frac{3}{4} \int_{x_0}^1 \chi |D_t^2 u^{n+1}|^2 dx \\
& \leq \frac{\mu}{4} \int_{x_0}^1 \chi \{ \rho^n(r^n)^4 |D_t^2 D_x u^{n+1}|^2 + \frac{2|D_t^2 u^{n+1}|^2}{\rho^n(r^n)^2} \} dx + \mu \int_{x_0}^1 \underbrace{\chi \left| \frac{D_t(\rho^n(r^n)^4)}{\rho^n(r^n)^4} \right|^2}_{\rho^n(r^n)^4} \rho^n(r^n)^4 |D_x u^{n+1}|^2 dx \\
& + \mu \int_{x_0}^1 \underbrace{\chi |\rho^n(r^n)^2 D_t(\frac{1}{\rho^n(r^n)^2})|^2}_{\frac{1}{\rho^n(r^n)^2}} \frac{2|u^{n+1}|^2}{\rho^n(r^n)^2} dx + \frac{\mu}{2} \int_{x_0}^1 \chi \underbrace{\frac{D_t(\rho^n(r^n)^4)}{\rho^n(r^n)^4}}_{\rho^n(r^n)^4} \rho^n(r^n)^4 |D_t D_x u^{n+1}|^2 dx \\
& + \frac{\mu}{2} \int_{x_0}^1 \chi \underbrace{\rho^n(r^n)^2 D_t(\frac{1}{\rho^n(r^n)^2})}_{\frac{1}{\rho^n(r^n)^2}} \frac{2|D_t u^{n+1}|^2}{\rho^n(r^n)^2} dx + \int_{x_0}^1 \chi |D_t((r^n)^2 D_x p^n)|^2 dx + \frac{1}{\mu} \int_{x_0}^1 \chi 16\pi^2 \rho^n(r^n)^2 x^2 \\
& |D_t(\frac{1}{(r^n)^2})|^2 dx - \mu \int_{x_0}^{x_1} \chi' \rho^n(r^n)^4 D_t D_x u^{n+1} D_t^2 u^{n+1} dx - \mu \int_{x_0}^{x_1} \chi' D_t(\rho^n(r^n)^4) D_x u^{n+1} D_t^2 u^{n+1} dx \\
& \quad \underline{\underline{}}
\end{aligned}$$

Corresponding Eulerian estimates for $\partial_t \partial_{r^n} u^{n+1}$ are given as follows:

$$\begin{aligned}
& \frac{\mu}{2} \frac{d}{dt} \int_0^{r_2-d} \zeta \{ |\partial_t \partial_{r^n} u^{n+1}|^2 + \frac{2|\partial_t u^{n+1}|^2}{(r^n)^2} \} (r^n)^2 dr^n + \frac{3}{4} \int_0^{r_2-d} \zeta \rho^n |\partial_t^2 u^{n+1}|^2 (r^n)^2 dr^n \\
& \leq 4 \int_0^{r_2-d} \zeta \underbrace{|\frac{\partial_t \rho^n}{\rho^n}|^2}_{\rho^n} \rho^n |\partial_t u^{n+1}|^2 (r^n)^2 dr^n + 8 \int_0^{r_2-d} \zeta \underbrace{|\rho^n D_t r^n|^2}_{\rho^n} |\partial_t \partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n \\
& \quad + 8 \int_0^{r_2-d} \zeta \underbrace{\frac{|\partial_t(\rho^n D_t r^n)|^2}{\rho^n}}_{\rho^n} |\partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n + 4 \int_0^{r_2-d} \zeta \frac{1}{\rho^n} |\partial_t \partial_{r^n} p^n|^2 (r^n)^2 dr^n \\
& \quad + 64\pi^2 \int_0^{r_2-d} \zeta \frac{|\partial_t(\rho^n x)|^2}{\rho^n (r^n)^2} dr^n - \mu \int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) (\partial_t \partial_{r^n} u^{n+1}) \partial_t^2 u^{n+1} (r^n)^2 dr^n
\end{aligned}$$

Similarly, we can perform the higher order energy estimates. Here is the estimate of $D_t^2 u^{n+1}$. Multiply $D_t^2(5.2)$ by $\chi D_t^2 u^{n+1}$ and integrate:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi |D_t^2 u^{n+1}|^2 dx + \frac{3\mu}{4} \int_{x_0}^1 \chi \{ \rho^n (r^n)^4 |D_t^2 D_x u^{n+1}|^2 + \frac{2|D_t^2 u^{n+1}|^2}{\rho^n (r^n)^2} \} dx \\
& \leq 12\mu \int_{x_0}^1 \chi \underbrace{|\frac{D_t(\rho^n (r^n)^4)}{\rho^n (r^n)^4}|^2}_{\rho^n} \rho^n (r^n)^4 |D_t D_x u^{n+1}|^2 dx + 3\mu \int_{x_0}^1 \chi \frac{|D_t^2(\rho^n (r^n)^4)|^2}{(\rho^n)^3 (r^n)^8} \underbrace{|\rho^n (r^n)^2 D_x u^{n+1}|^2}_{\rho^n} dx \\
& \quad + 32\mu \int_{x_0}^1 \chi \underbrace{|\rho^n (r^n)^2 D_t(\frac{1}{\rho^n (r^n)^2})|^2}_{\rho^n} \frac{|D_t u^{n+1}|^2}{\rho^n (r^n)^2} dx + 8\mu \int_{x_0}^1 \chi \rho^n (r^n)^2 |D_t^2(\frac{1}{\rho^n (r^n)^2})|^2 \underbrace{|u^{n+1}|^2}_{\rho^n} dx \\
& \quad + \frac{3}{\mu} \int_{x_0}^1 \chi \frac{|D_t^2((r^n)^2 p^n)|^2}{\rho^n (r^n)^4} dx + \frac{2}{\mu} \int_{x_0}^1 \chi \rho^n (r^n)^2 |D_t^2(\frac{2p^n}{\rho^n r^n})|^2 dx + \frac{32\pi^2}{\mu} \int_{x_0}^1 \chi \rho^n (r^n)^2 |D_t^2(\frac{x}{(r^n)^2})|^2 dx \\
& \quad - \mu \int_{x_0}^{x_1} \chi' \rho^n (r^n)^4 D_t^2 D_x u^{n+1} D_t^2 u^{n+1} dx - 2\mu \int_{x_0}^{x_1} \chi' D_t(\rho^n (r^n)^4) D_t D_x u^{n+1} D_t^2 u^{n+1} dx \\
& \quad - \mu \int_{x_0}^{x_1} \chi' D_t^2(\rho^n (r^n)^4) D_x u^{n+1} D_t^2 u^{n+1} dx + \int_{x_0}^{x_1} \chi' D_t^2((r^n)^2 p^n) D_t^2 u^{n+1} dx
\end{aligned}$$

The Eulerian estimate of $\partial_t^2 u^{n+1}$: multiply $\partial_t^2(5.4)$ by $\zeta \rho^n \partial_t^2 u^{n+1} (r^n)^2$ and integrate:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^{r_2-d} \zeta \rho^n |\partial_t^2 u^{n+1}|^2 (r^n)^2 dr^n + \frac{3\mu}{4} \int_0^{r_2-d} \chi \{ |\partial_t^2 \partial_{r^n} u^{n+1}|^2 + \frac{2|\partial_t^2 u^{n+1}|^2}{(r^n)^2} \} (r^n)^2 dr^n \\
& \leq -\frac{3}{2} \int_0^{r_2-d} \zeta \underbrace{\frac{\partial_t \rho^n}{\rho^n}}_{\rho^n} \rho^n |\partial_t^2 u^{n+1}|^2 (r^n)^2 dr^n + \frac{2}{\mu} \int_0^{r_2-d} \zeta \underbrace{|\rho^n D_t r^n|^2}_{\rho^n} |\partial_t^2 u^{n+1}|^2 (r^n)^2 dr^n \\
& \quad + \int_0^{r_2-d} \zeta \underbrace{|\partial_t u^{n+1}|^2}_{\rho^n} \rho^n |\partial_t^2 u^{n+1}|^2 (r^n)^2 dr^n + \int_0^{r_2-d} \underbrace{|\partial_{r^n} u^{n+1}|^2}_{\rho^n} \rho^n |\partial_t^2 u^{n+1}|^2 (r^n)^2 dr^n \\
& \quad + \int_0^{r_2-d} \zeta \frac{(\partial_t^2 \rho^n)^2}{\rho^n} (r^n)^2 dr^n + \frac{10}{\mu} \int_0^{r_2-d} \zeta |\partial_t^2 p^n|^2 (r^n)^2 dr^n + \frac{16\pi^2}{\mu} \int_0^{r_2-d} \zeta |\partial_t^2(\rho^n x)|^2 dr^n \\
& \quad + \int_0^{r_2-d} \zeta \frac{|\partial_t^2(\rho^n D_t r^n)|^2}{\rho^n} (r^n)^2 dr^n + \frac{4}{\mu} \int_0^{r_2-d} \zeta \underbrace{\frac{|\partial_t(\rho^n D_t r^n)|^2}{\rho^n}}_{\rho^n} \rho^n |\partial_t^2 u^{n+1}|^2 (r^n)^2 dr^n \\
& \quad - \mu \int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) \partial_t^2 u^{n+1} \partial_t^2 \partial_{r^n} u^{n+1} (r^n)^2 dr^n + \int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) \partial_t^2 p^n \partial_t^2 u^{n+1} (r^n)^2 dr^n
\end{aligned}$$

Now we turn into $D_t^2 D_x u^{n+1}$. Multiply $D_t^2(5.2)$ by $\chi D_t^3 u^{n+1}$ and integrate:

$$\begin{aligned}
& \frac{\mu}{2} \frac{d}{dt} \int_{x_0}^1 \chi \{ \rho^n(r^n)^4 |D_t^2 D_x u^{n+1}|^2 + \frac{2|D_t^2 u^{n+1}|^2}{\rho^n(r^n)^2} \} dx + \frac{3}{4} \int_{x_0}^1 \chi |D_t^3 u^{n+1}|^2 dx \\
& \leq \frac{\mu}{4} \int_{x_0}^1 \chi \{ \rho^n(r^n)^4 |D_t^3 D_x u^{n+1}|^2 + \frac{2|D_t^3 u^{n+1}|^2}{\rho^n(r^n)^2} \} dx + \frac{\mu}{2} \int_{x_0}^1 \underbrace{\chi \frac{D_t(\rho^n(r^n)^4)}{\rho^n(r^n)^4} \rho^n(r^n)^4}_{\chi \frac{D_t(\rho^n(r^n)^4)}{\rho^n(r^n)^4}} |D_t^2 D_x u^{n+1}|^2 dx \\
& \quad + \frac{\mu}{2} \int_{x_0}^1 \underbrace{\chi \rho^n(r^n)^2 D_t \left(\frac{1}{\rho^n(r^n)^2} \right)}_{\chi \rho^n(r^n)^2 D_t \left(\frac{1}{\rho^n(r^n)^2} \right)} \frac{2|D_t^2 u^{n+1}|^2}{\rho^n(r^n)^2} dx + 8\mu \int_{x_0}^1 \underbrace{\chi \left| \frac{D_t(\rho^n(r^n)^4)}{\rho^n(r^n)^4} \right|^2}_{\chi \left| \frac{D_t(\rho^n(r^n)^4)}{\rho^n(r^n)^4} \right|^2} \rho^n(r^n)^4 |D_t D_x u^{n+1}|^2 dx \\
& \quad + 2\mu \int_{x_0}^1 \chi \frac{|D_t^2(\rho^n(r^n)^4)|^2}{(\rho^n)^3(r^n)^8} \underbrace{|\rho^n(r^n)^2 D_x u^{n+1}|^2}_{|\rho^n(r^n)^2 D_x u^{n+1}|^2} dx + 4\mu \int_{x_0}^1 \underbrace{\chi \left| \rho^n(r^n)^2 D_t \left(\frac{1}{\rho^n(r^n)^2} \right) \right|^2}_{\chi \left| \rho^n(r^n)^2 D_t \left(\frac{1}{\rho^n(r^n)^2} \right) \right|^2} \frac{2|D_t u^{n+1}|^2}{\rho^n(r^n)^2} dx \\
& \quad + 8\mu \int_{x_0}^1 \chi \rho^n(r^n)^2 |D_t^2 \left(\frac{1}{\rho^n(r^n)^2} \right)|^2 \underbrace{|u^{n+1}|^2}_{|u^{n+1}|^2} dx + 2 \int_{x_0}^1 \chi |D_t^2((r^n)^2 D_x p^n)|^2 dx \\
& \quad + 32\pi^2 \int_{x_0}^1 \chi x^2 |D_t^2 \left(\frac{1}{(r^n)^2} \right)|^2 dx - \mu \int_{x_0}^{x_1} \chi' \rho^n(r^n)^4 D_t^2 D_x u^{n+1} D_t^3 u^{n+1} dx \\
& \quad - 2\mu \int_{x_0}^{x_1} \chi' D_t(\rho^n(r^n)^4) D_t D_x u^{n+1} D_t^3 u^{n+1} dx - \mu \int_{x_0}^{x_1} \chi' D_t^2(\rho^n(r^n)^4) D_x u^{n+1} D_t^3 u^{n+1} dx \\
& \quad \underline{\underline{\hspace{10cm}}}
\end{aligned}$$

For $\partial_t^2 \partial_{r^n} u^{n+1}$, multiply $\partial_t^2(5.4)$ by $\zeta \rho^n \partial_t^3 u^{n+1} (r^n)^2$ and integrate:

$$\begin{aligned}
& \frac{\mu}{2} \frac{d}{dt} \int_0^{r_2-d} \zeta \{ |\partial_t^2 \partial_{r^n} u^{n+1}|^2 + \frac{2|\partial_t^2 u^{n+1}|^2}{(r^n)^2} \} (r^n)^2 dr^n + \frac{1}{2} \int_0^{r_2-d} \zeta \rho^n |\partial_t^3 u^{n+1}|^2 (r^n)^2 dr^n \\
& \leq 16 \int_0^{r_2-d} \underbrace{\zeta \left| \frac{\partial_t \rho^n}{\rho^n} \right|^2}_{\zeta \left| \frac{\partial_t \rho^n}{\rho^n} \right|^2} \rho^n |\partial_t^2 u^{n+1}|^2 (r^n)^2 dr^n + 4 \int_0^{r_2-d} \zeta \frac{|\partial_t^2 \rho^n|^2}{\rho^n} \underbrace{|\partial_t u^{n+1}|^2}_{|\partial_t u^{n+1}|^2} (r^n)^2 dr^n \\
& + 8 \int_0^{r_2-d} \zeta \underbrace{\rho^n |D_t r^n|^2}_{\rho^n |D_t r^n|^2} |\partial_t^2 \partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n + 32 \int_0^{r_2-d} \zeta \underbrace{\frac{|\partial_t(\rho^n D_t r^n)|^2}{\rho^n}}_{\frac{|\partial_t(\rho^n D_t r^n)|^2}{\rho^n}} |\partial_t \partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n \\
& + 8 \int_0^{r_2-d} \zeta \frac{|\partial_t^2(\rho^n D_t r^n)|^2}{\rho^n} \underbrace{|\partial_{r^n} u^{n+1}|^2}_{|\partial_{r^n} u^{n+1}|^2} (r^n)^2 dr^n + 2 \int_0^{r_2-d} \zeta \frac{|\partial_t^2 \partial_{r^n} p^n|^2}{\rho^n} (r^n)^2 dr^n \\
& + 32\pi^2 \int_0^{r_2-d} \zeta \frac{|\partial_t^2(\rho^n x)|^2}{\rho^n (r^n)^2} dr^n - \mu \int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) (\partial_t^2 \partial_{r^n} u^{n+1}) (\partial_t^3 u^{n+1}) (r^n)^2 dr^n \\
& \underline{\underline{\hspace{10cm}}}
\end{aligned}$$

Here is the estimate of the full time derivative terms. Multiply $D_t^3(5.2)$ by $\chi D_t^3 u^{n+1}$ and integrate to

get:

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi |D_t^3 u^{n+1}|^2 dx + \frac{3\mu}{4} \int_{x_0}^1 \chi \{ \rho^n (r^n)^4 |D_t^3 D_x u^{n+1}|^2 + \frac{2|D_t^3 u^{n+1}|^2}{\rho^n (r^n)^2} \} dx \leq 36\mu \int_{x_0}^1 \chi \cdot \\
& \underbrace{\left| \frac{D_t(\rho^n (r^n)^4)}{\rho^n (r^n)^4} \right|^2 \rho^n (r^n)^4 |D_t^2 D_x u^{n+1}|^2 dx}_{\text{}} + 36\mu \int_{x_0}^1 \chi \frac{|D_t^2(\rho^n (r^n)^4)|^2}{(\rho^n)^3 (r^n)^8} \underbrace{|\rho^n (r^n)^2 D_t D_x u^{n+1}|^2}_{\text{}} dx \\
& + 4\mu \int_{x_0}^1 \chi \frac{|D_t^3(\rho^n (r^n)^4)|^2}{(\rho^n)^3 (r^n)^8} \underbrace{|\rho^n (r^n)^2 D_x u^{n+1}|^2}_{\text{}} dx + \frac{4}{\mu} \int_{x_0}^1 \chi \frac{|D_t^3((r^n)^2 p^n)|^2}{\rho^n (r^n)^4} dx \\
& + 90\mu \int_{x_0}^1 \chi \underbrace{|\rho^n (r^n)^2 D_t(\frac{1}{\rho^n (r^n)^2})|^2}_{\text{}} \frac{|D_t^2 u^{n+1}|^2}{\rho^n (r^n)^2} dx + 90\mu \int_{x_0}^1 \chi \rho^n (r^n)^2 |D_t^2(\frac{1}{\rho^n (r^n)^2})|^2 \underbrace{|D_t u^{n+1}|^2}_{\text{}} dx \\
& + 10\mu \int_{x_0}^1 \chi \rho^n (r^n)^2 |D_t^3(\frac{1}{\rho^n (r^n)^2})|^2 \underbrace{|u^{n+1}|^2}_{\text{}} dx + \frac{5}{2\mu} \int_{x_0}^1 \chi \rho^n (r^n)^2 |D_t^3(\frac{2p^n}{\rho^n r^n})|^2 dx \\
& + \frac{40\pi^2}{\mu} \int_{x_0}^1 \chi \rho^n (r^n)^2 x^2 |D_t^3(\frac{1}{(r^n)^2})|^2 dx - \mu \int_{x_0}^{x_1} \chi' \rho^n (r^n)^4 D_t^3 D_x u^{n+1} D_t^3 u^{n+1} dx \\
& - 3\mu \int_{x_0}^{x_1} \chi' D_t(\rho^n (r^n)^4) D_t^2 D_x u^{n+1} D_t^3 u^{n+1} dx - 3\mu \int_{x_0}^{x_1} \chi' D_t^2(\rho^n (r^n)^4) D_t D_x u^{n+1} D_t^3 u^{n+1} dx \\
& \underline{\underline{-\mu \int_{x_0}^{x_1} \chi' D_t^3(\rho^n (r^n)^4) D_x u^{n+1} D_t^3 u^{n+1} dx + \int_{x_0}^{x_1} \chi' D_t^3((r^n)^2 p^n) D_t^3 u^{n+1} dx}}
\end{aligned}$$

Finally, we are ready to the Eulerian estimate of $\partial_t^3 u^{n+1}$. In order to control the nonlinear terms involving higher derivatives, displaying three-dimensional feature, we employ the Gagliardo-Nirenberg inequality. Multiply $\partial_t^3(5.4)$ by $\zeta \rho^n \partial_t^3 u^{n+1} (r^n)^2$ and integrate to get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^{r_2-d} \zeta \rho^n |\partial_t^3 u^{n+1}|^2 (r^n)^2 dr^n + \frac{\mu}{2} \int_0^{r_2-d} \zeta \{ |\partial_t^3 \partial_{r^n} u^{n+1}|^2 + \frac{2|\partial_t^3 u^{n+1}|^2}{(r^n)^2} \} (r^n)^2 dr^n \\
& \leq -\frac{5}{2} \int_0^{r_2-d} \zeta \underbrace{\frac{\partial_t \rho^n}{\rho^n}} \rho^n |\partial_t^3 u^{n+1}|^2 (r^n)^2 dr^n + \frac{2}{\mu} \int_0^{r_2-d} \zeta \underbrace{\rho^n |D_t r^n|^2}_{\text{}} \rho^n |\partial_t^3 u^{n+1}|^2 (r^n)^2 dr^n \\
& + 36 \int_0^{r_2-d} \zeta \underbrace{\frac{|\partial_t(\rho^n D_t r^n)|^2}{\rho^n}} |\partial_t^2 \partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n + 4 \int_0^{r_2-d} \zeta \frac{|\partial_t^3(\rho^n D_t r^n)|^2}{\rho^n} \underbrace{|\partial_{r^n} u^{n+1}|^2}_{\text{}} (r^n)^2 dr^n \\
& + \frac{1}{8} \int_0^{r_2-d} \zeta \rho^n |\partial_t^3 u^{n+1}|^2 (r^n)^2 dr^n + 4 \int_0^{r_2-d} \zeta \frac{|\partial_t^3 \rho^n|^2}{\rho^n} \underbrace{|\partial_t u^{n+1}|^2}_{\text{}} (r^n)^2 dr^n \\
& + \frac{6}{\mu} \int_0^{r_2-d} \zeta |\partial_t^3 p^n|^2 (r^n)^2 dr^n + \frac{64\pi^2}{\mu} \int_0^{r_2-d} \zeta |\partial_t^3(\rho^n x)|^2 dr^n \\
& \underbrace{-3 \int_0^{r_2-d} \zeta \partial_t^2 \rho^n \partial_t^2 u^{n+1} \partial_t^3 u^{n+1} (r^n)^2 dr^n + 3 \int_0^{r_2-d} \zeta \partial_t^2(\rho^n D_t r^n) \partial_t \partial_{r^n} u^{n+1} \partial_t^3 u^{n+1} (r^n)^2 dr^n}_{\text{**}} \\
& - \mu \int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) (\partial_t^3 u^{n+1}) (\partial_t^3 \partial_{r^n} u^{n+1}) (r^n)^2 dr^n + \int_{r_1+d}^{r_2-d} (\partial_{r^n} \zeta) (\partial_t^3 p^n) (\partial_t^3 u^{n+1}) (r^n)^2 dr^n
\end{aligned}$$

For $\star\star$, first by the Cauchy-Schwarz inequality, we write it as

$$\begin{aligned} \star\star \leq & \frac{1}{8} \int_0^{r_2-d} \zeta \rho^n |\partial_t^3 u^{n+1}|^2 (r^n)^2 dr^n + 6 \int_0^{r_2-d} \zeta \left| \frac{\partial_t^2 \rho^n}{\sqrt{\rho^n}} \right|^4 (r^n)^2 dr^n + 6 \int_0^{r_2-d} \zeta |\partial_t^2 u^{n+1}|^4 (r^n)^2 dr^n \\ & + 6 \int_0^{r_2-d} \zeta \left| \frac{\partial_t^2 (\rho^n D_t r^n)}{\sqrt{\rho^n}} \right|^4 (r^n)^2 dr^n + 6 \int_0^{r_2-d} \zeta |\partial_t \partial_{r^n} u^{n+1}|^4 (r^n)^2 dr^n \end{aligned}$$

The first term will be absorbed into the LHS and we apply the Gagliardo-Nirenberg inequality

$$\|f\|_{L^4} \leq \frac{1}{2} \|\nabla f\|_{L^2}^{\frac{3}{4}} \|f\|_{L^2}^{\frac{1}{4}} \quad \text{in } \mathbb{R}^3$$

to the rest of terms. For instance, we have the following:

$$\begin{aligned} \int_0^{r_2-d} \zeta \left| \frac{\partial_t^2 \rho^n}{\sqrt{\rho^n}} \right|^4 (r^n)^2 dr^n & \leq \left(\int_0^{r_2-d} \zeta |\partial_{r^n} \left(\frac{\partial_t^2 \rho^n}{\sqrt{\rho^n}} \right)|^2 (r^n)^2 dr^n \right)^{\frac{3}{2}} \left(\int_0^{r_2-d} \zeta \left| \frac{\partial_t^2 \rho^n}{\sqrt{\rho^n}} \right|^2 (r^n)^2 dr^n \right)^{\frac{1}{2}} \\ \int_0^{r_2-d} \zeta |\partial_t \partial_{r^n} u^{n+1}|^4 (r^n)^2 dr^n & \leq \left(\int_0^{r_2-d} \zeta |\partial_t \partial_{r^n}^2 u^{n+1}|^2 (r^n)^2 dr^n \right)^{\frac{3}{2}} \left(\int_0^{r_2-d} \zeta |\partial_t \partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n \right)^{\frac{1}{2}} \end{aligned}$$

Now we combine all the estimates that we have obtained so far to get:

$$\begin{aligned} \frac{d}{dt} \mathcal{F}^{n+1}(t) + \mathcal{D}^{n+1}(t) & \leq \frac{7}{8} \mathcal{D}^{n+1}(t) + C \sup \{ \underbrace{\text{under-braced terms}} \} |\mathcal{F}^{n+1}(t)| \\ & \quad + C(\mathcal{F}^{n+1}(t))^2 + \mathcal{G}^{n+1}(t) + \underline{\underline{\{ \text{double underlined terms} \}}} \end{aligned}$$

Next, we list the terms of energy type of the previous n-th step: $\mathcal{G}^{n+1}(t)$.

$$\begin{aligned} & \sum_{i=0}^3 \int_{x_0}^1 \chi \frac{|D_t^i((r^n)^2 p^n)|^2}{\rho^n (r^n)^4} dx, \sum_{i=0}^3 \int_{x_0}^1 \chi \rho^n (r^n)^2 |D_t^i \left(\frac{2p^n}{\rho^n r^n} \right)|^2 dx, \sum_{i=0}^3 \int_{x_0}^1 \chi \rho^n (r^n)^2 |D_t^i \left(\frac{x}{(r^n)^2} \right)|^2 dx, \\ & \sum_{j=0}^2 \int_{x_0}^1 \chi |D_t^j((r^n)^2 D_x p^n)|^2 dx; \sum_{i=0}^3 \int \zeta |\partial_t^i p^n|^2 (r^n)^2 dr^n, \sum_{j=0}^2 \int \zeta \frac{|\partial_t^j \partial_{r^n} p^n|^2}{\rho^n} (r^n)^2 dr^n, \\ & \sum_{i=0}^3 \int \zeta |\partial_t^i (x \rho^n)|^2 dr^n, \sum_{j=0}^2 \int \zeta \frac{|\partial_t^j (x \rho^n)|^2}{\rho^n (r^n)^2} dr^n, \sum_{i=1}^3 \int \zeta \frac{|\partial_t^i (\rho^n D_t r^n)|^2}{\rho^n} (r^n)^2 dr^n, \\ & \sum_{j=2}^3 \int \zeta \frac{|\partial_t^j \rho^n|^2}{\rho^n} (r^n)^2 dr^n, \left(\int_0^{r_2-d} \zeta |\partial_{r^n} \left(\frac{\partial_t^2 \rho^n}{\sqrt{\rho^n}} \right)|^2 (r^n)^2 dr^n \right)^{\frac{3}{2}} \left(\int_0^{r_2-d} \zeta \left| \frac{\partial_t^2 \rho^n}{\sqrt{\rho^n}} \right|^2 (r^n)^2 dr^n \right)^{\frac{1}{2}}, \\ & \left(\int_0^{r_2-d} \zeta |\partial_{r^n} \left(\frac{\partial_t^2 (\rho^n D_t r^n)}{\sqrt{\rho^n}} \right)|^2 (r^n)^2 dr^n \right)^{\frac{3}{2}} \left(\int_0^{r_2-d} \zeta \left| \frac{\partial_t^2 (\rho^n D_t r^n)}{\sqrt{\rho^n}} \right|^2 (r^n)^2 dr^n \right)^{\frac{1}{2}}. \end{aligned}$$

Claim 1. There exists a constant C_4 so that

$$\mathcal{G}^{n+1}(t) \leq C_4 \{ (1 + M^{n+1} + |M^n|^2 + \mathcal{F}^n(t)) \mathcal{H}^n(t) + (\mathcal{H}^n(t))^2 \} + C_5 e^{C_6(M^n + M^{n-1})T} \mathcal{F}^n(t).$$

Proof of Claim 1: Recall (6.1). And note that the dynamics of r^n follows from ρ^n :

$$\begin{aligned}
D_t r^n &= -\frac{1}{(r^n)^2} \int_0^x \frac{D_t \rho^n}{(\rho^n)^2} dy; \\
D_t^2 r^n &= -\frac{2|D_t r^n|^2}{r^n} - \frac{1}{(r^n)^2} \int_0^x \frac{D_t^2 \rho^n}{(\rho^n)^2} dy + \frac{2}{(r^n)^2} \int_0^x \frac{(D_t \rho^n)^2}{(\rho^n)^3} dy; \\
D_t^3 r^n &= -\frac{4D_t^2 r^n D_t r^n}{r^n} + 2r^n \left(\frac{D_t r^n}{r^n}\right)^3 + \frac{2D_t r^n}{(r^n)^3} \int_0^x \frac{D_t \rho^n}{(\rho^n)^2} dy - \frac{1}{(r^n)^2} \int_0^x \frac{D_t^3 \rho^n}{(\rho^n)^2} dy \\
&\quad + \frac{6}{(r^n)^2} \int_0^x \frac{D_t^2 \rho^n D_t \rho^n}{(\rho^n)^3} dy - \frac{4D_t r^n}{(r^n)^3} \int_0^x \frac{(D_t \rho^n)^2}{(\rho^n)^3} dy - \frac{6}{(r^n)^2} \int_0^x \frac{(D_t \rho^n)^3}{(\rho^n)^4} dy.
\end{aligned}$$

Because of (6.1) and

$$\begin{aligned}
\left| \int_0^x \frac{D_t^2 \rho^n}{(\rho^n)^2} dy \right| &\leq \left(\int_0^x \frac{|D_t^2 \rho^n|^2}{|\rho^n|^3} dy \right)^{\frac{1}{2}} \left(\int_0^x \frac{1}{\rho^n} dy \right)^{\frac{1}{2}} \leq C e^{(M^n + M^{n-1}) \frac{T}{2}} (\mathcal{F}^n)^{\frac{1}{2}} (r^n)^{\frac{3}{2}}, \\
\left| \int_0^x \frac{D_t^3 \rho^n}{(\rho^n)^2} dy \right| &\leq C e^{(M^n + M^{n-1}) \frac{T}{2}} (\mathcal{F}^n)^{\frac{1}{2}} (r^n)^{\frac{3}{2}},
\end{aligned}$$

one gets the following:

$$|D_t^2 r^n| \leq C |M^n|^2 r^n + C e^{(M^n + M^{n-1}) \frac{T}{2}} \frac{(\mathcal{F}^n)^{\frac{1}{2}}}{(r^n)^{\frac{1}{2}}}, \quad |D_t^3 r^n| \leq C |D_t^2 r^n|.$$

Now let us look at Lagrangian terms. Here we provide the details for the first and the last terms in the list. The first one deals with pure t -derivative terms.

$$\begin{aligned}
&\bullet \int_{x_0}^1 \chi (\rho^n)^{2\gamma-1} dx \leq C_{in} e^{\gamma M^n T} \int_{x_0}^1 \chi (\rho^n)^{\gamma-1} dx \leq C_{in} e^{\gamma M^n T} \mathcal{H}^n(t) \\
&\bullet \int_{x_0}^1 \chi \frac{|D_t((r^n)^2 (\rho^n)^\gamma)|^2}{\rho^n (r^n)^4} dx \leq 2 \int_{x_0}^1 \chi \left| \frac{D_t r^n}{r^n} \right|^2 (\rho^n)^{2\gamma-1} dx + 2\gamma^2 \int_{x_0}^1 \chi \left| \frac{D_t \rho^n}{\rho^n} \right|^2 (\rho^n)^{2\gamma-1} dx \\
&\quad \leq C_{in} |M^n|^2 e^{\gamma M^n T} \mathcal{H}^n(t) \\
&\bullet \int_{x_0}^1 \chi \frac{|D_t^2((r^n)^2 (\rho^n)^\gamma)|^2}{\rho^n (r^n)^4} dx \leq C \int_{x_0}^1 \chi \left\{ \left| \frac{D_t^2 r^n}{r^n} \right|^2 + \left| \frac{D_t r^n}{r^n} \right|^2 \left| \frac{D_t \rho^n}{\rho^n} \right|^2 + \left| \frac{D_t \rho^n}{\rho^n} \right|^4 \right\} (\rho^n)^{2\gamma-1} dx \\
&\quad + C \int_{x_0}^1 \chi \rho^{2\gamma} \frac{|D_t^2 \rho^n|^2}{(\rho^n)^3} dx \\
&\quad \leq C_{in} (|M^n|^2 + |M^n|^4 + e^{(2M^n + M^{n-1})T} \frac{\mathcal{F}^n(t)}{(r_0)^3}) e^{\gamma M^n T} \mathcal{H}^n(t) \\
&\quad + C_{in} e^{(M^n + M^{n-1})T} e^{2\gamma M^n T} \mathcal{F}^n(t) \\
&\bullet \int_{x_0}^1 \chi \frac{|D_t^3((r^n)^2 (\rho^n)^\gamma)|^2}{\rho^n (r^n)^4} dx \leq C_{in} (|M^n|^2 + |M^n|^4 + e^{(2M^n + M^{n-1})T} \frac{\mathcal{F}^n(t)}{(r_0)^3}) e^{\gamma M^n T} \mathcal{H}^n(t) \\
&\quad + C_{in} e^{(M^n + M^{n-1})T} e^{2\gamma M^n T} \mathcal{F}^n(t)
\end{aligned}$$

The last one treats x -derivative terms. Due to the dynamics of r^n , we obtain the following:

$$\begin{aligned}
& \bullet \int_{x_0}^1 \chi |(r^n)^2 D_x p^n|^2 dx = A^2 \gamma^2 \int_{x_0}^1 \chi (\rho^n)^{2\gamma-2} (r^n)^4 |D_x \rho^n|^2 \leq C \mathcal{H}^n(t) \\
& \bullet \int_{x_0}^1 \chi |D_t((r^n)^2 D_x p^n)|^2 dx \leq C(1 + |M^n|^2) \mathcal{H}^n(t) \\
& \bullet \int_{x_0}^1 \chi |D_t^2((r^n)^2 D_x p^n)|^2 dx \leq C(1 + |M^n|^2 + |M^n|^4 + e^{(2M^n + M^{n-1})T} \frac{\mathcal{F}^n(t)}{(r_0)^3}) \mathcal{H}^n(t)
\end{aligned}$$

For Eulerian terms, first note that

$$\partial_{r^n} = \frac{\rho^n (r^n)^2}{\rho^{n-1} (r^{n-1})^2} \partial_{r^{n-1}}, \quad \rho^n (r^n)^2 dr^n = \rho^{n-1} (r^{n-1})^2 dr^{n-1}.$$

Hence, by change of variables, we obtain, for instance

$$\begin{aligned}
& \bullet \int \zeta |p^n|^2 (r^n)^2 dr^n = A^2 \gamma^2 \int \zeta (\rho^n)^{2\gamma-1} \rho^{n-1} (r^{n-1})^2 dr^{n-1} \leq C_{in} e^{((\gamma-1)M^n + M^{n-1})T} \mathcal{H}^n(t) \\
& \bullet \int \zeta \frac{|\partial_{r^n} p^n|^2}{\rho^n} (r^n)^2 dr^n = A^2 \gamma^2 \int \zeta \frac{(\rho^n)^\gamma}{\rho^{n-1}} \left| \frac{r^n}{r^{n-1}} \right|^4 (\rho^n)^{\gamma-2} |\partial_{r^{n-1}} \rho^n|^2 (r^{n-1})^2 dr^{n-1} \\
& \quad \leq C_{in} e^{(\frac{3\gamma+4}{3}M^n + \frac{7}{3}M^{n-1})T} \mathcal{H}^n(t) \\
& \bullet \left(\int_0^{r_2-d} \zeta |\partial_{r^n} \left(\frac{\partial_t^2 \rho^n}{\sqrt{\rho^n}} \right)|^2 (r^n)^2 dr^n \right)^{\frac{3}{2}} \left(\int_0^{r_2-d} \zeta \left| \frac{\partial_t^2 \rho^n}{\sqrt{\rho^n}} \right|^2 (r^n)^2 dr^n \right)^{\frac{1}{2}} \leq C_{in} e^{C(M^n + M^{n-1})T} (\mathcal{H}^n(t))^2
\end{aligned}$$

Other t -derivative terms can be estimated in the same way. This finishes the proof of Claim 1.

Here are the terms, under-braced in the above estimates, needed to be point-wisely estimated (L^∞) to close the above energy estimates:

$$\begin{aligned}
& \text{Lagrangian terms}(x_0 \leq x \leq 1) \Rightarrow \left| \frac{D_t \rho^n}{\rho^n} \right|, \left| \frac{D_t r^n}{r^n} \right|, |u^{n+1}|, |\rho^n (r^n)^2 D_x u^{n+1}|, |\rho^n (r^n)^2 D_t D_x u^{n+1}| \\
& \text{Eulerian terms}(0 \leq r \leq r_2 - d) \Rightarrow \left| \frac{\partial_t \rho^n}{\rho^n} \right|, \rho^n |D_t r^n|^2, \frac{|\partial_t (\rho^n D_t r^n)|^2}{\rho^n}, |u^{n+1}|, |\partial_{r^n} u^{n+1}|, |\partial_t u^{n+1}|
\end{aligned}$$

As we noted in (6.1), any ρ^n , r^n related terms can be bounded by \mathcal{F}^n and \mathcal{H}^n . We need to show that other u^{n+1} related terms can be bounded by \mathcal{F}^{n+1} as well as \mathcal{F}^n , \mathcal{H}^n . Now let us estimate M^{n+1} :

$$\begin{aligned}
& \sup |\rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}| \leq \underbrace{\int_{x_0}^1 \chi |\rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}| dx}_{(i)} \\
& \quad + \underbrace{\int_{x_0}^1 \chi |D_x (\rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n})| dx}_{(ii)} + \left(\int_0^{r_2-d} \zeta |\partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n}|^2 (r^n)^2 dr^n \right)^{\frac{1}{2}} \\
& \quad + \left(\int_0^{r_2-d} \zeta |\nabla_n (\partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n})|^2 (r^n)^2 dr^n \right)^{\frac{1}{2}} + \left(\int_0^{r_2-d} \zeta |\nabla_n^2 (\partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n})|^2 (r^n)^2 dr^n \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
(i) &\leq C(\int_{x_0}^1 \chi \rho^n dx)^{\frac{1}{2}} (\int_{x_0}^1 \chi \{\rho^n (r^n)^4 |D_x u^{n+1}|^2 + \frac{2|u^{n+1}|^2}{\rho^n (r^n)^2}\} dx)^{\frac{1}{2}} \\
(ii) &= \int_{x_0}^1 \chi \left| \frac{D_t u^{n+1}}{(r^n)^2} + D_x p^n + \frac{4\pi x}{(r^n)^4} \right| dx \\
&\leq (\int_{x_0}^1 \chi \frac{1}{(r^n)^4} dx)^{\frac{1}{2}} \{ (\int_{x_0}^1 \chi |D_t u^{n+1}|^2 dx)^{\frac{1}{2}} + (\int_{x_0}^1 \chi (r^n)^4 |D_x p^n|^2 dx)^{\frac{1}{2}} \} \\
&\quad + (\int_{x_0}^1 \chi \frac{1}{\rho^n (r^n)^4} dx)^{\frac{1}{2}} (\int_{x_0}^1 \chi \frac{16\pi^2 x^2 \rho^n}{(r^n)^4} dx)^{\frac{1}{2}}
\end{aligned}$$

Hence,

$$\begin{aligned}
M^{n+1} &\leq C e^{\frac{M^n}{2}T} (\int_{x_0}^1 \chi \rho_{in} dx)^{\frac{1}{2}} (\mathcal{F}^{n+1})^{\frac{1}{2}} + e^{\frac{2M^n}{3}T} (\int_{x_0}^1 \chi \frac{1}{r_{in}^4} dx)^{\frac{1}{2}} \{ (\mathcal{F}^{n+1})^{\frac{1}{2}} + (\mathcal{H}^n)^{\frac{1}{2}} \} \\
&\quad + e^{\frac{7M^n}{6}T} (\int_{x_0}^1 \chi \frac{1}{\rho_{in} r_{in}^4} dx)^{\frac{1}{2}} (\mathcal{H}^n)^{\frac{1}{2}} + C e^{\frac{M^n}{2}T} \sup_{0 \leq r \leq r_2-d} \left| \frac{1}{\sqrt{\rho_{in}}} \right| (\mathcal{F}^{n+1})^{\frac{1}{2}}
\end{aligned} \tag{6.6}$$

Next we estimate L^∞ bound of $|\rho^n (r^n)^2 D_x u^{n+1}| (= |\partial_{r^n} u^{n+1}|)$:

$$\begin{aligned}
\sup |\rho^n (r^n)^2 D_x u^{n+1}| &\leq \int_{x_0}^1 \chi |\rho^n (r^n)^2 D_x u^{n+1}| dx + \int_{x_0}^1 \chi |D_x (\rho^n (r^n)^2 D_x u^{n+1})| dx \\
&\quad + (\int_0^{r_2-d} \zeta |\partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n)^{\frac{1}{2}} + (\int_0^{r_2-d} \zeta |\nabla_n \partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n)^{\frac{1}{2}} \\
&\quad + (\int_0^{r_2-d} \zeta |\nabla_n^2 \partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n)^{\frac{1}{2}} \\
&\leq (\int_{x_0}^1 \chi \rho^n dx)^{\frac{1}{2}} (\int_{x_0}^1 \chi \rho^n (r^n)^4 |D_x u^{n+1}|^2 dx)^{\frac{1}{2}} + \int_{x_0}^1 \chi \left| \frac{D_t u^{n+1}}{(r^n)^2} + D_x p^n + \frac{4\pi x}{(r^n)^4} - \underbrace{D_x \left(\frac{2u^{n+1}}{r^n} \right)}_{\star} \right| dx \\
&\quad + (\int_0^{r_2-d} \zeta |\partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n)^{\frac{1}{2}} + (\int_0^{r_2-d} \zeta |\nabla_n \partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n)^{\frac{1}{2}} \\
&\quad + (\int_0^{r_2-d} \zeta |\nabla_n^2 \partial_{r^n} u^{n+1}|^2 (r^n)^2 dr^n)^{\frac{1}{2}}
\end{aligned}$$

For \star , first separate the integral into two by the quotient rule, and use Hölder's inequality.

$$\begin{aligned}
\star &\leq \int_{x_0}^1 \chi \left| \frac{D_x u^{n+1}}{r^n} \right| dx + \int_{x_0}^1 \chi \left| \frac{u^{n+1}}{\rho^n (r^n)^4} \right| dx \\
&\leq (\int_{x_0}^1 \chi \rho^n (r^n)^4 |D_x u^{n+1}|^2 dx)^{\frac{1}{2}} (\int_{x_0}^1 \chi \frac{1}{\rho^n (r^n)^6} dx)^{\frac{1}{2}} + (\int_{x_0}^1 \chi \frac{|u^{n+1}|^2}{\rho^n (r^n)^2} dx)^{\frac{1}{2}} (\int_{x_0}^1 \chi \frac{1}{\rho^n (r^n)^6} dx)^{\frac{1}{2}}
\end{aligned}$$

Hence, we get

$$\begin{aligned}
|\rho^n (r^n)^2 D_x u^{n+1}| &\leq C e^{\frac{M^n}{2}T} (\int_{x_0}^1 \chi \rho_{in} dx)^{\frac{1}{2}} (\mathcal{F}^{n+1})^{\frac{1}{2}} + e^{\frac{2M^n}{3}T} (\int_{x_0}^1 \chi \frac{1}{r_{in}^4} dx)^{\frac{1}{2}} \{ (\mathcal{F}^{n+1})^{\frac{1}{2}} + (\mathcal{H}^n)^{\frac{1}{2}} \} \\
&\quad + e^{\frac{7M^n}{6}T} (\int_{x_0}^1 \chi \frac{1}{\rho_{in} r_{in}^4} dx)^{\frac{1}{2}} (\mathcal{H}^n)^{\frac{1}{2}} + e^{\frac{3M^n}{2}T} (\int_{x_0}^1 \chi \frac{1}{\rho_{in} r_{in}^6} dx)^{\frac{1}{2}} (\mathcal{F}^{n+1})^{\frac{1}{2}} \\
&\quad + C e^{\frac{M^n}{2}T} \sup_{0 \leq r \leq r_2-d} \left| \frac{1}{\sqrt{\rho_{in}}} \right| (\mathcal{F}^{n+1})^{\frac{1}{2}}
\end{aligned}$$

Similarly, L^∞ bound of other terms can be obtained. The estimate of \mathcal{F}^{n+1} indeed can be closed by $\mathcal{F}^{n+1}, \mathcal{F}^n, \mathcal{H}^n$. It completes (6.2). \square

Next we move onto \mathcal{H}^{n+1} .

Proof. of Lemma 6.3: Note that the point-wise estimates of ρ^{n+1} and r^{n+1} , as in (6.1), can be obtained by using M^{n+1} :

$$\begin{aligned}\rho_{in}e^{-M^{n+1}T} &\leq \rho^{n+1} \leq \rho_{in}e^{M^{n+1}T}; \quad r_{in}e^{-\frac{M^{n+1}T}{3}} \leq r^{n+1} \leq r_{in}e^{\frac{M^{n+1}T}{3}}; \\ \left| \frac{D_t \rho^{n+1}}{\rho^{n+1}} \right| &= |\rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}| \leq M^{n+1}; \\ D_t r^{n+1} &= -\frac{1}{(r^{n+1})^2} \int_0^x \frac{D_t \rho^{n+1}}{\rho^{n+1}} \frac{1}{\rho^{n+1}} dy \Rightarrow \left| \frac{D_t r^{n+1}}{r^{n+1}} \right| \leq \frac{M^{n+1}}{3}.\end{aligned}$$

Now we are ready to construct (6.5). Observe that we need all the mixed derivatives estimates both in Lagrangian and in Eulerian formulations because of overlapping terms, and note that the overlapping region holds one-dimensional feature. Before we derive Eulerian energy estimates, we complete L^2 -type of Lagrangian estimates, in particular of spatial derivatives of ρ^{n+1} . In order to do so, we turn to the continuity equation (5.6). First, here is the zeroth order estimate. Multiply (5.6) by $(\rho^{n+1})^{\gamma-2}$ and integrate:

$$\begin{aligned}&\frac{1}{\gamma-1} \frac{d}{dt} \int_{x_0}^1 \chi (\rho^{n+1})^{\gamma-1} dx + \int_{x_0}^1 \chi (\rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}) (\rho^{n+1})^{\gamma-1} dx = 0 \\ \Rightarrow &\frac{1}{\gamma-1} \frac{d}{dt} \int_{x_0}^1 \chi (\rho^{n+1})^{\gamma-1} dx \leq M^{n+1} \int_{x_0}^1 \chi (\rho^{n+1})^{\gamma-1} dx\end{aligned}$$

To estimate $D_x \rho^{n+1}$, take D_x of (5.6), multiply by $(\rho^{n+1})^\alpha (r^n)^4 D_x \rho^{n+1}$, and integrate:

$$\begin{aligned}&\frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi (\rho^{n+1})^\alpha (r^n)^4 |D_x \rho^{n+1}|^2 dx - \frac{\alpha+2}{2} \int_{x_0}^1 \chi (\rho^{n+1})^{\alpha-1} D_t \rho^{n+1} (r^n)^4 |D_x \rho^{n+1}|^2 dx \\ &\quad - 2 \int_{x_0}^1 \chi (\rho^{n+1})^\alpha D_t r^n (r^n)^3 |D_x \rho^{n+1}|^2 dx \\ &\quad + \int_{x_0}^1 \chi (\rho^{n+1})^{\alpha+1} (r^n)^4 D_x (\rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}) D_x \rho^{n+1} dx = 0 \\ \Rightarrow &\frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi (\rho^{n+1})^\alpha (r^n)^4 |D_x \rho^{n+1}|^2 dx \leq \frac{(\alpha+2)M^{n+1} + 2M^n + \epsilon}{2} \int_{x_0}^1 \chi (\rho^{n+1})^\alpha (r^n)^4 |D_x \rho^{n+1}|^2 dx \\ &\quad + \frac{1}{2\epsilon} \int_{x_0}^1 \chi (\rho^{n+1})^{\alpha+2} \left| (r^n)^2 D_x (\rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n}) \right|^2 dx \\ &\quad = \frac{1}{\mu} \{ D_t u^{n+1} + (r^n)^2 D_x p^n + \frac{4\pi x}{(r^n)^2} \} \\ &\leq C(M^{n+1} + M^n) \mathcal{H}^{n+1}(t) + C_{in} e^{CM^{n+1}T} (\mathcal{F}^{n+1}(t) + \mathcal{H}^n(t))\end{aligned}$$

We may choose $\alpha = 2\gamma - 2$. t -derivatives do not destroy the structure of the equation and thus estimates of $D_t D_x \rho^{n+1}$, $D_t^2 D_x \rho^{n+1}$ follow in the similar fashion with the same weight. Now we take one more spatial derivative:

$$\begin{aligned}D_t D_x^2 \rho^{n+1} &= -\frac{1}{\mu} \left\{ \frac{D_t D_x u^{n+1}}{(r^n)^2} - \frac{2D_t u^{n+1}}{\rho^n (r^n)^5} + D_x^2 p^n + \frac{4\pi}{(r^n)^4} - \frac{16\pi x}{\rho^n (r^n)^7} \right\} \rho^{n+1} \\ &\quad - \frac{2}{\mu} \left\{ \frac{D_t u^{n+1}}{(r^n)^2} + D_x p^n + \frac{4\pi x}{(r^n)^4} \right\} D_x \rho^{n+1} - \left\{ \rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n} \right\} D_x^2 \rho^{n+1}\end{aligned}$$

Multiply by $(\rho^{n+1})^{\alpha_1}(r^n)^8 D_x^2 \rho^{n+1}$ and integrate:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi(\rho^{n+1})^{\alpha_1} (r^n)^8 |D_x^2 \rho^{n+1}|^2 dx &= \frac{\alpha_1 + 2}{2} \int_{x_0}^1 \chi(\rho^{n+1})^{\alpha_1-1} D_t \rho^{n+1} (r^n)^8 |D_x^2 \rho^{n+1}|^2 dx \\ &\quad + 4 \int_{x_0}^1 \chi(\rho^{n+1})^{\alpha_1} D_t r^n (r^n)^7 |D_x^2 \rho^{n+1}|^2 dx \\ - \frac{1}{\mu} \int_{x_0}^1 \chi \{ (r^n)^2 D_t D_x u^{n+1} - \frac{2 D_t u^{n+1}}{\rho^n r^n} + (r^n)^4 D_x^2 p^n + 4\pi - \frac{16\pi x}{\rho^n (r^n)^3} \} &(\rho^{n+1})^{\alpha_1+1} (r^n)^4 D_x^2 \rho^{n+1} dx \\ - \frac{2}{\mu} \int_{x_0}^1 \chi \{ D_t u^{n+1} + (r^n)^2 D_x p^n + \frac{4\pi x}{(r^n)^2} \} &(r^n)^2 D_x \rho^{n+1} (\rho^{n+1})^{\alpha_1} (r^n)^4 D_x^2 \rho^{n+1} dx \end{aligned}$$

Let

$$K^{n+1} \equiv \sup_{x_0 \leq x \leq 1} |(\rho^{n+1})^{\frac{\alpha_1}{2}} (r^n)^2 D_x \rho^{n+1}|.$$

Apply the Sobolev embedding theorem and the Cauchy-Schwarz inequality to get

$$\begin{aligned} K^{n+1} &\leq \int_{x_0}^1 |(\rho^{n+1})^{\frac{\alpha_1}{2}} (r^n)^2 D_x \rho^{n+1}| dx + \int_{x_0}^1 |D_x ((\rho^{n+1})^{\frac{\alpha_1}{2}} (r^n)^2 D_x \rho^{n+1})| dx \\ &\leq \left(\int_{x_0}^1 (\rho^{n+1})^{\alpha_1} (r^n)^4 |D_x \rho^{n+1}|^2 dx \right)^{\frac{1}{2}} + \int_{x_0}^1 (\rho^{n+1})^{\frac{\alpha_1}{2}-1} (r^n)^2 |D_x \rho^{n+1}|^2 dx \\ &\quad + 2 \int_{x_0}^1 \frac{(\rho^{n+1})^{\frac{\alpha_1}{2}}}{\rho^n r^n} |D_x \rho^{n+1}| dx + \left(\int_{x_0}^1 (\rho^{n+1})^{\alpha_1} (r^n)^4 |D_x^2 \rho^{n+1}|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Note that $|\frac{\rho^n}{\rho^{n+1}}|$ or $|\frac{\rho^{n+1}}{\rho^n}| \leq e^{(M^{n+1}+M^n)T}$. Hence, we may choose $\frac{\alpha_1}{2} - 1 = 2\gamma - 2$, and with this α_1 , we get

$$K^{n+1} \leq (1 + C_{in} e^{\gamma M^{n+1}T}) (\mathcal{H}^{n+1}(t))^{\frac{1}{2}}$$

and also the following:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{x_0}^1 \chi(\rho^{n+1})^{\alpha_1} (r^n)^8 |D_x^2 \rho^{n+1}|^2 dx &\leq \left(\frac{\alpha_1 + 2}{2} M^{n+1} + 2M^n + \epsilon \right) \int_{x_0}^1 \chi(\rho^{n+1})^{\alpha_1} (r^n)^8 |D_x^2 \rho^{n+1}|^2 dx \\ &\quad + \frac{1}{2\mu\epsilon} \int_{x_0}^1 \chi(\rho^{n+1})^{\alpha_1+2} | (r^n)^2 D_t D_x u^{n+1} - \frac{2 D_t u^{n+1}}{\rho^n r^n} + (r^n)^4 D_x^2 p^n + 4\pi - \frac{16\pi x}{\rho^n (r^n)^3} |^2 dx \\ &\quad + \frac{4(K^{n+1})^2}{\mu\epsilon} \int_{x_0}^1 \chi | D_t u^{n+1} + (r^n)^2 D_x p^n + \frac{4\pi x}{(r^n)^2} |^2 dx \\ &\leq C(M^{n+1} + M^n) \mathcal{H}^{n+1}(t) + (C_{in} e^{CM^{n+1}T} + \mathcal{H}^{n+1}(t)) (\mathcal{F}^{n+1}(t) + \mathcal{H}^n(t)) \end{aligned}$$

$D_t D_x^2 \rho^{n+1}$ can be estimated in the same way with the same weight. Take one more D_x to get

$$\begin{aligned} D_t D_x^3 \rho^{n+1} &= -\frac{1}{\mu} \left\{ \frac{D_t D_x^2 u^{n+1}}{(r^n)^2} - \frac{4 D_t D_x u^{n+1}}{\rho^n (r^n)^5} - 2 D_t u^{n+1} D_x \left(\frac{1}{\rho^n (r^n)^5} \right) + D_x^3 p^n \right. \\ &\quad + D_x \left(\frac{4\pi}{(r^n)^4} - \frac{16\pi x}{\rho^n (r^n)^7} \right) \} \rho^{n+1} - \frac{3}{\mu} \left\{ \frac{D_t D_x u^{n+1}}{(r^n)^2} - \frac{2 D_t u^{n+1}}{\rho^n (r^n)^5} + D_x^2 p^n + \frac{4\pi}{(r^n)^4} - \frac{16\pi x}{\rho^n (r^n)^7} \right\} D_x \rho^{n+1} \\ &\quad - \frac{3}{\mu} \left\{ \frac{D_t u^{n+1}}{(r^n)^2} + D_x p^n + \frac{4\pi x}{(r^n)^4} \right\} D_x^2 \rho^{n+1} - \left\{ \rho^n (r^n)^2 D_x u^{n+1} + \frac{2u^{n+1}}{r^n} \right\} D_x^3 \rho^{n+1} \end{aligned}$$

Now consider weights $(\rho^{n+1})^{\alpha_2} (r^n)^{12}$. The problematic term is

$$\int_{x_0}^1 \chi(\rho^{n+1})^{\alpha_2} (r^n)^{12} D_x^2 \rho^n D_x \rho^{n+1} D_x^3 \rho^{n+1} dx:$$

$$\begin{aligned} & \underbrace{A\gamma \int_{x_0}^1 \chi(\rho^{n+1})^{\alpha_2} (\rho^n)^{\gamma-1} (r^n)^{12} D_x^2 \rho^n D_x \rho^{n+1} D_x^3 \rho^{n+1} dx}_{(1)} \\ & + A\gamma(\gamma-1) \underbrace{\int_{x_0}^1 \chi(\rho^{n+1})^{\alpha_2} (\rho^n)^{\gamma-2} (r^n)^{12} (D_x \rho^n)^2 D_x \rho^{n+1} D_x^3 \rho^{n+1} dx}_{(2)} \end{aligned}$$

We estimate (1) and (2) respectively as follows:

$$\begin{aligned} (1) & \leq K^{n+1} \int_{x_0}^1 \chi(\rho^{n+1})^{\frac{\alpha_2-\alpha_1}{2}} (\rho^n)^{\gamma-1} (r^n)^4 |D_x^2 \rho^n| (\rho^{n+1})^{\frac{\alpha_2}{2}} (r^n)^6 |D_x^3 \rho^{n+1}| dx \\ & \Rightarrow \frac{\alpha_2 - \alpha_1}{2} + \gamma - 1 \geq \frac{\alpha_1}{2} \\ (2) & \leq K^{n+1} K^n \int_{x_0}^1 \chi(\rho^{n+1})^{\frac{\alpha_2-\alpha_1}{2}} (\rho^n)^{\gamma-2-\frac{\alpha_1}{2}} (r^n)^2 |D_x \rho^n| (\rho^{n+1})^{\frac{\alpha_2}{2}} (r^n)^6 |D_x^3 \rho^{n+1}| dx \\ & \Rightarrow \frac{\alpha_2 - \alpha_1}{2} + \gamma - 2 - \frac{\alpha_1}{2} \geq \frac{\alpha}{2} \end{aligned}$$

Hence, we choose α_2 as follows: $\alpha_2 \geq 2\alpha_1 + \alpha + 4 - 2\gamma \geq 2(4\gamma - 2) + 2\gamma - 2 + 4 - 2\gamma = 8\gamma - 2$. On the other hand, in Eulerian coordinates, we use the approximate equation (5.7) to get L^2 -type of estimates. First, here is the zeroth order estimate:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \zeta (\rho^{n+1})^{\beta+2} (r^n)^2 dr^n - \frac{\beta}{2} \int \zeta \partial_t \rho^{n+1} (\rho^{n+1})^{\beta+1} (r^n)^2 dr^n \\ & + \underbrace{\int \zeta D_t r^n (\rho^{n+1})^{\beta+1} \partial_{r^n} \rho^{n+1} (r^n)^2 dr^n}_{\star} + \int \zeta (\rho^{n+1})^{\beta+2} \left\{ \partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n} \right\} (r^n)^2 dr^n = 0 \end{aligned}$$

For \star :

$$\begin{aligned} (\beta+2) \cdot \star & = - \int \partial_{r^n} \zeta D_t r^n (\rho^{n+1})^{\beta+2} (r^n)^2 dr^n - \int \zeta \partial_{r^n} (D_t r^n) (\rho^{n+1})^{\beta+2} (r^n)^2 dr^n \\ & \quad - \int \zeta D_t r^n (\rho^{n+1})^{\beta+2} 2r^n dr^n \end{aligned}$$

Note that $D_t r^n$ is more or less a zeroth order term in the following sense:

$$\begin{aligned} \partial_{r^n} (D_t r^n) & = \frac{2}{(r^n)^3} \int_0^x \frac{D_t \rho^n}{(\rho^n)^2} dy - \frac{1}{(r^n)^2} \frac{D_t \rho^n}{(\rho^n)^2} \rho^n (r^n)^2 = -\frac{2D_t r^n}{r^n} - \frac{D_t \rho^n}{\rho^n} \\ \partial_t (D_t r^n) & = \frac{1}{(r^n)^2} \frac{D_t \rho^n}{(\rho^n)^2} \rho^n (r^n)^2 D_t r^n = \frac{D_t \rho^n}{\rho^n} D_t r^n \end{aligned}$$

Let ∂ be either temporal or spatial derivative ($\partial_t \rho^{n+1} = D_t \rho^{n+1} - D_t r^n \partial_{r^n} \rho^{n+1}$),

$$\begin{aligned} & \partial_t \partial \rho^{n+1} + \partial (D_t r^n) \partial_{r^n} \rho^{n+1} + D_t r^n \partial_{r^n} \partial \rho^{n+1} + \partial \rho^{n+1} \left\{ \partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n} \right\} \\ & + \rho^{n+1} \partial \left\{ \partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n} \right\} = 0 \end{aligned}$$

Multiply by $(\rho^{n+1})^\beta \partial \rho^{n+1} (r^n)^2$ and integrate it to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \zeta(\rho^{n+1})^\beta |\partial \rho^{n+1}|^2 (r^n)^2 dr^n - \frac{\beta}{2} \int \zeta(\rho^{n+1})^{\beta-1} \partial_t \rho^{n+1} |\partial \rho^{n+1}|^2 (r^n)^2 dr^n + \int \zeta(\rho^{n+1})^\beta \cdot \\ & \partial(D_t r^n) \partial_{r^n} \rho^{n+1} \partial \rho^{n+1} (r^n)^2 dr^n + \int \zeta(\rho^{n+1})^{\beta+1} \partial \rho^{n+1} D_t r^n \partial_{r^n} \partial \rho^{n+1} (r^n)^2 dr^n + \int \zeta(\rho^{n+1})^\beta \cdot \\ & \underbrace{\{\partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n}\} |\partial \rho^{n+1}|^2 (r^n)^2 dr^n + \int \zeta(\rho^{n+1})^{\beta+1} \partial \{\partial_{r^n} u^{n+1} + \frac{2u^{n+1}}{r^n}\} \partial \rho^{n+1} (r^n)^2 dr^n}_{\star} = 0 \end{aligned}$$

For \star , we get

$$\begin{aligned} 2 \cdot \star = & - \int \partial_{r^n} \zeta(\rho^{n+1})^\beta D_t r^n |\partial \rho^{n+1}|^2 (r^n)^2 dr^n - \beta \int \zeta(\rho^{n+1})^{\beta-1} \partial_{r^n} \rho^{n+1} D_t r^n |\partial \rho^{n+1}|^2 (r^n)^2 dr^n \\ & - \int \zeta(\rho^{n+1})^\beta \partial_{r^n} (D_t r^n) |\partial \rho^{n+1}|^2 (r^n)^2 dr^n - \int \zeta(\rho^{n+1})^\beta D_t r^n |\partial \rho^{n+1}|^2 2r^n dr^n \end{aligned}$$

We may choose $\beta = \gamma - 2$. Similarly, the energy estimates of higher order derivatives of ρ^{n+1} can be performed. Note that $\partial^3 \rho^{n+1}$, namely up to the third derivatives, needs to be estimated in order to close the energy estimates. We may choose the same weights $\beta = \gamma - 2$. We refer Lemma 4.1 for closing the estimates.

Now, based on the above estimates, let us examine ρ^{n+1} integrals with respect to dr^{n+1} , which we used at the previous steps. By the definition of r^n and r^{n+1} , $\rho^n (r^n)^2 dr^n = \rho^{n+1} (r^{n+1})^2 dr^{n+1}$ and thus $(r^{n+1})^2 dr^{n+1} = \frac{\rho^n}{\rho^{n+1}} (r^n)^2 dr^n$. Now we want to show that dr^{n+1} integrals can be controlled by dr^n integrals derived as above. This can be done easily because

$$\int \zeta(\rho^{n+1})^{2\gamma} (r^{n+1})^2 dr^{n+1} = \int \zeta(\rho^{n+1})^{2\gamma} \frac{\rho^n}{\rho^{n+1}} (r^n)^2 dr^n.$$

Note that $|\frac{\rho^n}{\rho^{n+1}}| \leq e^{(M^{n+1}+M^n)T}$. This completes the proof of Lemma 6.3. \square

Proof. of Theorem 1.2: By Proposition 6.1, now we can take $n \rightarrow \infty$ and we get limits $u(t, x) = u(t, r)$, $\rho(t, x) = \rho(t, r)$, $r(t, x)$ as well as the lagrangian transformation $x = \int_0^r \rho s^2 ds$; $r(t, x) = (3 \int_0^x \frac{1}{\rho} dy)^{\frac{1}{3}}$; $D_t r(t, x) = u(t, x)$. Furthermore, those limits serve the solution to the problem. Since $\mathcal{E}(t) \sim \lim_{n \rightarrow \infty} \{\mathcal{F}^n(t) + \mathcal{H}^n(t)\}$, the energy bound is easily obtained. Now it remains to show the uniqueness. Let (ρ_1, u_1, r_1) and (ρ_2, u_2, r_2) be two strong solutions to (1.12) satisfying the same initial conditions. Note that it is enough to show that $u_1 = u_2$, since it right away implies $r_1 = r_2$ from the dynamics of r : $D_t r = u$ and thus $\rho_1 = \rho_2$ as well. Now let us consider the momentum equations for u_1 and u_2 in Lagrangian coordinates:

$$\begin{aligned} D_t u_1 - \mu D_x (\rho_1 r_1^4 D_x u_1) + \mu \frac{2u_1}{\rho_1 r_1^2} &= -r_1^2 D_x p_1 - \frac{4\pi x}{r_1^2}; \\ D_t u_2 - \mu D_x (\rho_2 r_2^4 D_x u_2) + \mu \frac{2u_2}{\rho_2 r_2^2} &= -r_2^2 D_x p_2 - \frac{4\pi x}{r_2^2}. \end{aligned}$$

Subtracting one from another, we get the equation for $u_1 - u_2$ as follows:

$$\begin{aligned} D_t (u_1 - u_2) - \mu D_x (\rho_1 r_1^4 D_x (u_1 - u_2)) + \mu \frac{2(u_1 - u_2)}{\rho_1 r_1^2} &= -r_1^2 D_x p_1 - \frac{4\pi x}{r_1^2} + r_2^2 D_x p_2 + \frac{4\pi x}{r_2^2} \\ &+ \mu D_x ((\rho_1 r_1^4 - \rho_2 r_2^4) D_x u_2) - 2\mu u_2 \left(\frac{1}{\rho_1 r_1^2} - \frac{1}{\rho_2 r_2^2} \right) \end{aligned}$$

Multiply by $u_1 - u_2$ and integrate to get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^1 |u_1 - u_2|^2 dx + \mu \int_0^1 \rho_1 r_1 |D_x(u_1 - u_2)|^2 + \frac{2|u_1 - u_2|^2}{\rho_1 r_1^2} dx \\
&= \int_0^1 (r_1^2 p_1 - r_2^2 p_2) D_x(u_1 - u_2) dx + \int_0^1 \left(\frac{2p_1}{\rho_1 r_1} - \frac{2p_2}{\rho_2 r_2} \right) (u_1 - u_2) dx - 4\pi \int_0^1 \left(\frac{x}{r_1^2} - \frac{x}{r_2^2} \right) (u_1 - u_2) dx \\
&\quad - \mu \int_0^1 (\rho_1 r_1^4 - \rho_2 r_2^4) D_x u_2 D_x(u_1 - u_2) dx - 2\mu \int_0^1 u_2 \left(\frac{1}{\rho_1 r_1^2} - \frac{1}{\rho_2 r_2^2} \right) (u_1 - u_2) dx \\
&\leq \underbrace{\left(\int_0^1 \frac{1}{\rho_1 r_1^4} |r_1^2 p_1 - r_2^2 p_2|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 \rho_1 r_1^4 |D_x(u_1 - u_2)|^2 dx \right)^{\frac{1}{2}}}_{(i)} \\
&\quad + \underbrace{\left\{ \left(2 \int_0^1 \rho_1 r_1^2 \left| \frac{p_1}{\rho_1 r_1} - \frac{p_2}{\rho_2 r_2} \right|^2 dx \right)^{\frac{1}{2}} + \left(8\pi^2 \int_0^1 \rho_1 r_1^2 x^2 \left| \frac{1}{r_1^2} - \frac{1}{r_2^2} \right|^2 dx \right)^{\frac{1}{2}} \right\} \left(\int_0^1 \frac{2|u_1 - u_2|^2}{\rho_1 r_1^2} dx \right)^{\frac{1}{2}}}_{(ii)} \\
&\quad + \underbrace{\mu \left(\int_0^1 \frac{1}{\rho_1 r_1^4} |(\rho_1 r_1^4 - \rho_2 r_2^4) D_x u_2|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 \rho_1 r_1^4 |D_x(u_1 - u_2)|^2 dx \right)^{\frac{1}{2}}}_{(iv)} \\
&\quad + \underbrace{\mu \left(2 \int_0^1 \rho_1 r_1^2 u_2^2 \left| \frac{1}{\rho_1 r_1^2} - \frac{1}{\rho_2 r_2^2} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 \frac{2|u_1 - u_2|^2}{\rho_1 r_1^2} dx \right)^{\frac{1}{2}}}_{(v)}
\end{aligned}$$

Now we would like to estimate (i) – (v) in terms of $u_1 - u_2$. In order to do so, we use the explicit formula for ρ_1, ρ_2 :

$$\rho_1(t, x) = \rho_{in}(x) e^{-\int_0^t \rho_1 r_1^2 D_x u_1 + \frac{2u_1}{r_1} d\tau}; \quad \rho_2(t, x) = \rho_{in}(x) e^{-\int_0^t \rho_2 r_2^2 D_x u_2 + \frac{2u_2}{r_2} d\tau}$$

Here we provide the detail for (iv) and other terms can be estimated in the same way. First (iv) can be bounded by

$$(iv) \leq M^2 \int_0^1 \frac{1}{\rho_1 r_1^4} \left| \frac{\rho_1 r_1^4 - \rho_2 r_2^4}{\rho_2 r_2^2} \right|^2 dx = M^2 \int_0^1 \frac{1}{\rho_1} \left| \frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2} \right|^2 dx \quad (6.7)$$

where M is the bound of $\rho_2 r_2^2 D_x u_2$. Note that

$$\begin{aligned}
\left| \frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2} \right|^2 &= \left| e^{\int_0^t \rho_2 r_2^2 D_x u_2 - \rho_1 r_1^2 D_x u_1 d\tau} - e^{\int_0^t \frac{2u_2}{r_2} - \frac{2u_1}{r_1} d\tau} \right|^2 \\
&\leq 2 \left| \int_0^t \rho_2 r_2^2 D_x u_2 - \rho_1 r_1^2 D_x u_1 d\tau - \int_0^t \frac{2u_2}{r_2} - \frac{2u_1}{r_1} d\tau \right|^2 \\
&\leq 4 \left| \int_0^t \rho_1 r_1^2 D_x(u_1 - u_2) d\tau \right|^2 + 4 \left| \int_0^t (\rho_1 r_1^2 - \rho_2 r_2^2) D_x u_2 d\tau \right|^2 \\
&\quad + 4 \left| \int_0^t \frac{2(u_1 - u_2)}{r_1} d\tau \right|^2 + 4 \left| \int_0^t 2u_2 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) d\tau \right|^2 \\
&\leq 4 \left(\int_0^t \rho_1 d\tau \right) \left(\int_0^t \rho_1 r_1^4 |D_x(u_1 - u_2)|^2 d\tau \right) + 4M^2 \left| \int_0^t \frac{\rho_1 r_1^2}{\rho_2 r_2^2} - 1 d\tau \right|^2 \\
&\quad + 4 \left(\int_0^t \rho_1 d\tau \right) \left(\int_0^t \frac{4|u_1 - u_2|^2}{\rho_1 r_1^2} d\tau \right) + 4M^2 \left| \int_0^t \frac{r_2}{r_1} - 1 d\tau \right|^2
\end{aligned}$$

Since

$$4M^2 \left| \int_0^t \frac{\rho_1 r_1^2}{\rho_2 r_2^2} - 1 d\tau \right|^2 \leq 8M^2 t \int_0^t \left| \frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2} \right|^2 d\tau + 8M^2 t \int_0^t \left| \frac{r_2^2}{r_1^2} - 1 \right|^2 d\tau,$$

$$\left| \frac{r_2}{r_1} + 1 \right| \leq 1 + e^{\frac{2M}{3}T} \quad \text{and} \quad 4M^2 \left| \int_0^t \frac{r_2}{r_1} - 1 d\tau \right|^2 \leq 4M^2 t \int_0^t \left| \frac{r_2}{r_1} - 1 \right|^2 d\tau,$$

we get, for $0 \leq t \leq T$ where T is sufficiently small to be fixed,

$$\left| \frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2} \right|^2 \leq 8M^2 T \int_0^t \left| \frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2} \right|^2 d\tau + 4M^2 T (3 + 4e^{\frac{4M}{3}T}) \int_0^t \left| \frac{r_2}{r_1} - 1 \right|^2 d\tau$$

$$+ 4 \left(\int_0^t \rho_1 d\tau \right) \left(\int_0^t \rho_1 r_1^4 |D_x(u_1 - u_2)|^2 + \frac{4|u_1 - u_2|^2}{\rho_1 r_1^2} d\tau \right).$$

On the other hand, in the same way, one can derive the similar inequality for $\left| \frac{r_2}{r_1} - 1 \right|^2$:

$$\left| \frac{r_2}{r_1} - 1 \right|^2 \leq 2 \left| \int_0^t \frac{u_2}{r_2} - \frac{u_1}{r_1} d\tau \right|^2 \leq 2M^2 T \int_0^t \left| \frac{r_2}{r_1} - 1 \right|^2 d\tau + 2 \left(\int_0^t \rho_1 d\tau \right) \left(\int_0^t \frac{|u_1 - u_2|^2}{\rho_1 r_1^2} d\tau \right)$$

By the Gronwall inequality, we get

$$\left| \frac{\rho_1 r_1^2}{\rho_2 r_2^2} - \frac{r_2^2}{r_1^2} \right|^2 + \left| \frac{r_2}{r_1} - 1 \right|^2 \leq 9 \left(\int_0^T \rho_1 d\tau \right) \left(\int_0^T \rho_1 r_1^4 |D_x(u_1 - u_2)|^2 + \frac{2|u_1 - u_2|^2}{\rho_1 r_1^2} d\tau \right) \cdot$$

$$(1 + 2M^2 T (7 + 8e^{\frac{4M}{3}T}) t e^{2M^2 T (7 + 8e^{\frac{4M}{3}T}) t})$$

Taking this into account (6.7), (iv) can be controlled as follows:

$$(iv) \leq C_{M,T} \int_0^1 \frac{1}{\rho_1} \left(\int_0^T \rho_1 d\tau \right) \left(\int_0^T \rho_1 r_1^4 |D_x(u_1 - u_2)|^2 + \frac{2|u_1 - u_2|^2}{\rho_1 r_1^2} d\tau \right) dx$$

$$\leq C_{M,T} \int_0^T \int_0^1 \rho_1 r_1^4 |D_x(u_1 - u_2)|^2 + \frac{2|u_1 - u_2|^2}{\rho_1 r_1^2} dx d\tau$$

where we have used $\left| \frac{1}{\rho_1} \int_0^T \rho_1 d\tau \right| \leq T e^{2MT}$. Following the same path, one can derive the following:

$$(i) + (ii) + (iii) + (iv) + (v) \leq C_{M,T} \int_0^T \int_0^1 \rho_1 r_1^4 |D_x(u_1 - u_2)|^2 + \frac{2|u_1 - u_2|^2}{\rho_1 r_1^2} dx d\tau$$

Hence, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^1 |u_1 - u_2|^2 dx + \frac{\mu}{2} \int_0^1 \rho_1 r_1 |D_x(u_1 - u_2)|^2 + \frac{2|u_1 - u_2|^2}{\rho_1 r_1^2} dx$$

$$\leq C_{M,T} \int_0^T \int_0^1 \rho_1 r_1^4 |D_x(u_1 - u_2)|^2 + \frac{2|u_1 - u_2|^2}{\rho_1 r_1^2} dx d\tau$$

Now we integrate over $[0, t]$ to get for $0 \leq t \leq T$ where T is sufficiently small,

$$\left\{ \frac{1}{2} \int_0^1 |u_1 - u_2|^2 dx \right\}(t) \leq \left\{ \frac{1}{2} \int_0^1 |u_1 - u_2|^2 dx \right\}(0),$$

and therefore the uniqueness follows. \square

Acknowledgments: The author would like to thank YAN GUO for many stimulating discussions. She also would like to give many thanks to CONSTANTINE DAFERMOS and CHAU-HSING SU for helpful

discussions.

References

- [1] Y. DENG, T.-P. LIU, T. YANG, Z.-A. YAO: Solutions of Euler-Poisson equations for gaseous stars, *Arch.Rational Mech.Anal.* **164** (2002), 261-285
- [2] B. DUCOMET, A. ZLOTNIK: Stabilization and stability for the spherically symmetric Navier-Stokes-Poisson system, *Appl.Math.Lett.* **18** (2005), 1190–1198
- [3] S. CHANDRASEKHAR: An introduction to the study of stellar structures, *University of Chicago Press*, 1938
- [4] L. EVANS: Partial Differential Equations, *Amer.Math.Soc.* **19** (1998)
- [5] J. JANG: Nonlinear instability in gravitational Euler-Poisson system for $\gamma = \frac{6}{5}$, *submitted to Arch.Rational Mech.Anal.*
- [6] S.-S. LIN: Stability of gaseous stars in spherically symmetric motions, *SIAM J.Math.Anal.* **28** (1997), 539-569
- [7] T. LUO, Z. XIN, T. YANG: Interface behavior of compressible Navier-Stokes equations with vacuum, *SIAM J.Math.Anal.* **31** (2000), 1175-1191
- [8] T. MAKINO: On a local existence theorem for the evolution equation of gaseous stars. Patterns and waves, *Stud.Math.Appl.* **18** (1986), 459-479
- [9] M. OKADA, T. MAKINO: Free boundary problem for the equation of spherically symmetric motion of viscous gas, *Japan J.Indust.Appl.Math.* **10** (1993), 219-235
- [10] S. MATUSU-NECASOVA, M. OKADA, T. MAKINO: Free boundary problem for the equation of spherically symmetric motion of viscous gas III, *Japan J.Indust.Appl.Math.* **14** (1997), 199-213